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# Three essays on fair division and decision making under uncertainty

by

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ABSTRACT

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The first chapter is based on a paper with Jin Li in fair division. It was recently discovered that on the domain of Leontief preferences, Hurwicz (1972)'s classic impossibility result does not hold; that is, one can find efficient, strategy-proof and individually rational rules to divide resources among agents. Here we consider the problem of dividing $l$ divisible goods among $n$ agents with the generalized Leontief preferences. We propose and characterize the class of generalized egalitarian rules which satisfy efficiency, group strategy-proofness, anonymity, resource monotonicity, population monotonicity, envy-freeness and consistency. On the Leontief domain, our rules generalize the egalitarian-equivalent rules with reference bundles. We also extend our rules to agent-specific and endowment-specific egalitarian rules. The former is a larger class of rules satisfying all the previous properties except anonymity and envy-freeness. The latter is a class of efficient, group strategy-proof, anonymous and individually rational rules when the resources are assumed to be privately owned.

The second and third chapters are based on two working papers of mine in decision making under uncertainty. In the second chapter, I study the wealth effect under uncertainty — how the wealth level impacts a decision maker's degree of uncertainty aversion. I axiomatize a class of preferences displaying decreasing absolute uncertainty aversion, which allows a decision maker to be more willing to take uncertainty-bearing behavior when he becomes wealthier. Three equivalent preference representations are obtained. The first is a variation on the constraint criterion of Hansen and Sargent (2001). The other two respectively generalize Gilboa and Schmeidler (1989)'s maxmin criterion and Maccheroni,

Marinacci and Rustichini (2006)'s variational representation. This class, when restricted to preferences exhibiting constant absolute uncertainty aversion, is exactly Maccheroni, Marinacci and Rustichini (2006)'s ariational preferences. Thus, the results further enable us to establish relationships among the representations for several important classes within variational preferences.

The three representations provide different decision rules to rationalize the same class of preferences. The three decision rules correspond to three ways which are proposed in the literature to identify a decision maker's perception about uncertainty and his attitude toward uncertainty. However, I give examples to show that these identifications conflict with each other. It means that there is much freedom in eliciting two unobservable and subjective factors, one's perception about and attitude toward uncertainty, from only his choice behavior. This exactly motivates the work in Chapter 3.

In the third chapter, I introduce confidence orders in addition to preference orders. Axioms are imposed on both orders to reveal a decision maker's perception about uncertainty and to characterize the following decision rule. A decision maker evaluates an act based on his aspiration and his confidence in this aspiration. Each act corresponds to a trade-off line between the two criteria: The more he aspires, the less his confidence in achieving the aspiration level. The decision maker ranks an act by the optimal combination of aspiration and confidence on its trade-off line according to an aggregating preference of his over the two-criterion plane. The aggregating preference indicates his uncertainty attitude, while his perception about uncertainty is summarized by a generalized second-order belief over the prior space, and this belief is revealed by his confidence order.

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# Chapter 1

# Egalitarian division under Leontief preferences

## 1.1 Introduction

In the fair division literature, efficiency and strategy-proofness typically imply totally unfair outcomes. For example, Zhou (1991) shows that an efficient and strategy-proof allocation rule must be dictatorial already in a two-agent economy with continuous, strictly monotonic and strictly convex preferences. Such negative result has been extended to several more restricted domains by many researchers. Serizawa and Weymark (2003) further shows that in a many-agent many-good economy no efficient and strategy-proof rule can guarantee every agent a consumption bundle bounded away from the origin. (Additional discussion of related literature is given at the end of this section.)

However, the picture changes a lot if we assume full complementarity among the goods and consider the domain of Leontief preferences. On the Leontief domain, for most efficient divisions of a given set of resources, some of the resources are redundant[1]. Thus, it makes sense to give the agents only the least amount of goods to achieve given welfare levels, while transferring the redundant resources to other potential users outside the rule[2]. We speak in this case of a *non-wasteful rule*. In addition to the normative concern of parsimony, the restriction to non-wasteful rules reduces the possibility of strategic manip-

[1]For example, in a two-agent two-good economy, both agents have the same preference represented by the utility function $u(x) = \min\{\frac{x_1}{2}, x_2\}$, and the endowment vector is (2,2). Then 1 unit of good 2 is redundant in any efficient allocation which divides up all the resources.

[2]Notice that here withholding the redundant resources does not affect efficiency since they are useless to the agents. It is different from the budget loss in VCG mechanisms which directly reduce the welfare of the agents.

ulation. It turns out that then there exist rules satisfying efficiency, strategy-proofness and many fairness axioms.

The Leontief preferences and the corresponding non-wasteful rules are of natural practical interests, as shown in the computer science literature like Ghodsi et al. (2010), Hindman et al. (2011), Bodwin et al. (2011), Joe-Wong et al. (2011), Dolev et al. (2012), etc. For example, they consider multiple resource sharing problems in cloud computing systems. The users are allocated with computing resources like CPU, memory and I/O resources to do their different jobs with heterogeneous demands. In such circumstance, each user needs the resources in a customized proportion while redundant resources should not be allocated in order to avoid waste.

Two earlier papers inspire our work. Ghodsi et al. (2010) are the first to propose non-wasteful rules for the Leontief domain. They prove that in a many-agent many-good economy the egalitarian-equivalent (EE) rule proposed by Pazner and Schmeidler (1978) (they call it the Dominant Resource Fairness mechanism) is efficient, strategyproof, envy-free and satisfies several other fairness axioms. Prior to them, Nicolò (2004) characterizes in a two-agent two-good economy with generalized Leontief preferences, a class of rules which are efficient, fully implementable in truthful strategies (a requirement stronger than strategy-proofness) and individually rational. However, Nicolò's rules are wasteful, and he finds it difficult to generalize his result to an economy with more agents and more goods.

Our contribution is to bring the existing results to a much more general level. Under Leontief preferences, we propose a class of non-wasteful rules which generalize the EE rules with reference bundles (see Section 3 for the relation of the EE rule and those with reference bundles). They satisfy efficiency, (group) strategy-proofness and almost all the fairness axioms in the literature (see below for further discussion). We also characterize our rules by these axioms. Moreover, the characterization works as well on a much larger preference domain — the generalized Leontief preference domain, which we shall discuss later. Lastly, we provide two natural extensions of our rules.

The rules we propose are called *generalized egalitarian rules* (defined in Section 3). A generalized egalitarian rule assumes that there is a continuous monotonic "benchmark preference" on the commodity space owned by the society. It looks for the non-wasteful efficient allocation where all the agents get the bundles among which the society is indifferent according to its benchmark preference. In another way, we can visualize that in the commodity space, the agents walk on their own "minimum-demand" paths associated with their Leontief preferences at some given speeds which guarantee that at any time they all simultaneously stand on the same indifference curve of the benchmark preference, and then our rule picks the end points where they reach the endowment feasibility constraints. Essentially, egalitarian rules set a standard for society to measure different ordinal preferences of the agents so that they are treated equally by this standard. While a classical EE rule makes the agents feel indifferent between their allocations and the same fraction of the social endowment, our rule gives the agents "equal" bundles according to a utility function of the society. It turns out that when the social endowment is fixed, a classic EE rule on the Leontief domain is one of our rules with a particular benchmark preference. We discuss about it in detail in Example 2 of Section 3.

There is another interpretation of generalized egalitarian rules. Thomson (1994) proposes a concept of equity to capture the notion of equal opportunities. Given a family $C$ of choice sets, he defines an equal opportunity allocation relative to $C$ as one giving every agent his optimal bundle from a common choice set in $C$. Since such an allocation is obtained by having the agents choose in a common choice set, they can be viewed to get equal opportunities. It turns out that a general egalitarian rule always picks the Pareto optimal equal opportunity allocation relative to a corresponding family of nested choice sets.

Our first main result (Theorem 1) shows that a generalized egalitarian rule satisfies efficiency, group strategy-proofness, anonymity, resource monotonicity, population monotonicity, envy-freeness and consistency; and conversely, given an efficient, resource monotonic and consistent rule, if it is either strategy-proof and anonymous, or envy-free, then it must be a generalized egalitarian rule.

All these axioms are very familiar in the fair division literature. Among the incentive compatibility axioms, group strategy-proofness is a very strong one. It allows no group of agents to misreport their preferences together and achieve Pareto improvement within the group (see Pattanaik (1978), Barberà (1979), Moulin and Shenker (2001), Juarez(2008)). For the fairness axioms, anonymity simply rules out the discrimination of the agents by their names; resource monotonicity guarantees that every agent benefits from the growth of the social endowment (see Roemer (1986a,b), Chun and Thomson (1988)); population monotonicity ensures that no agent will get worse off when less agents join in the division (see Thomson (1983)); and envy-freeness makes every agent weakly prefer his own allocation to anybody else's (see Foley (1967), Varian (1974, 1976)). The consistency axiom has also played an important role in the fair division literature, in particular, the rationing (or bankruptcy) problems (see Aumann and Maschler (1985), Young (1987), Thomson (1988)). It requires that when some agents leave first with their allocated bundles, if we apply the rule again to the reduced economy, the rest of the agents will still be allocated with the same bundles as in the original economy. For a survey of these and some other axioms in the fair division literature, see Thomson (2010).

Many of the axioms above are known to be very demanding and typically incompatible. For example, Moulin and Thomson (1988) show that any efficient and resource monotonic rule must generate envy in an economy with continuous, monotonic, convex and homothetic preferences. However, generalized egalitarian rules under Leontief preferences surprisingly satisfy them all.

Our rules and characterization apply for a much larger preference domain — the domain of generalized Leontief preferences (see Theorem 2). While for a standard Leontief preference, the set of minimum commodity bundles that achieve given utility levels, which we called the *critical set*, is a ray from the origin in the commodity space, the critical set of a generalized Leontief preference can be an arbitrary strictly increasing curve starting from the origin. In real life, generalized Leontief preferences are relevant when the agents are production units and the goods are inputs. For example, a group of people are dividing some cotton, silk and lace to make clothes. They would like to use these materials in different proportions according to their own tastes. Given the precise combination of the materials to make some pieces of clothes, more material of one kind is useless, which captures the essence of a Leontief preference. Moreover, when the amount of all materials increases, one might be able to make a dress instead of a shirt which requires different proportion of materials. There might also exist different types of returns to scale which alter the input proportion. Hence, one's critical set is an increasing curve, as exhibited in the generalized Leontief preferences.

Our results crucially depend on the restriction to non-wasteful rules. We give an example in Section 3 showing that our results do not hold without this restriction. Our characterization is tight with respect to all the axioms.

Our next two results (Theorem 3 and 4) extend the generalized egalitarian rules in two directions. First, instead of using one single benchmark preference to measure all agents' utilities, a rule may assign to each agent a personal welfare index and equalize their utilities according to these agent-specific welfare indices. This family of rules is a much larger and non-anonymous class. Naturally, we do not expect envy-freeness in this case. However, all the other good properties are preserved.

The second extension is motivated when the resources are assumed to be privately, rather than commonly, owned by the agents. A compelling requirement here is the voluntary participation of the agents in the social reallocation. This is ensured by the individual rationality axiom, which requires the allocation to an agent to be no worse (for this agent) than his initial endowment. In this case, we can set the welfare indices such that it is always an "equal treatment" allocation to give every agent the minimum bundle that provides him the same welfare level as his private endowment. The welfare indices then depend on the endowment profile. By slightly modifying the argument in Moulin and Thomson (1988), one can check that efficiency, resource monotonicity and individual rationality are also incompatible in our context. We show that our endowment-specific egalitarian rules are efficient, group strategy-proof, anonymous, consistent and individually rational.

For both agent-specific and endowment-specific rules, our results are one-sided and we leave the characterizations as open questions.

After the literature review below, the paper is organized as follows. Section 2 presents the basic model and the axioms. Section 3 defines the generalized egalitarian rules under Leontief preferences and gives the characterization result. Section 4 introduces the generalized Leontief preference domain, on which the characterization still holds. Section 5 contains the main proofs. Section 6 checks the tightness of our characterization. Section 7 and 8 provide two extensions of the generalized egalitarian rules: agent-specific and endowment-specific egalitarian rules. Section 9 provides concluding remarks. The appendix contains some supporting proofs.

**Related Literature**

For the incompatibility of efficiency and strategy-proofness with fairness properties in exchange economies, Hurwicz (1972) first proves that any efficient and individually rational rule is manipulable in two-agent, two-good economies where both agents have continuous, strictly convex, and strictly monotonic preferences. Dasgupta et al. (1979) replace individual rationality with non-dictatorship, while allowing discontinuous preferences. Zhou (1991) shows that in two-agent many-good exchange economies with the same preference domain as in Hurwicz (1972), a strategy-proof and efficient rule has to be inverse-dictatorial[3], and hence dictatorial. From then on, many authors consider various restricted domains, either obtain similar impossibility results or compromise with weakened axioms, such as Schummer (1997, 2004), Ju (2003), Hashimoto (2008), and Momi (2011a) for two-agent cases, Barberà and Jackson (1995), Kato and Ohseto (2002, 2004), Amorós (2002), Serizawa (2002), Serizawa and Weymark (2003), Ju (2004), Morimoto et al. (2010) and Momi (2011b) for many-agent cases. As we mentioned before, both Nicolò (2004) and Ghodsi et al. (2010) study the Leontief preference domain and achieve positive results. The main difference between their works is that Nicolò (2004) studies a two-agent two-good economy with generalized Leontief preferences and gives a characterization, while Ghodsi et al. (2010) study a many-agent many-good economy with standard Leontief preferences and give several one-sided results. In this paper, we consider generalized Leontief preferences and get very positive characterization results for many-agent many-good economy, without weakening any axioms. We would also like to mention that there is a large part of literature studying allocation rules for economies with public goods, such as Hurwicz and Walker (1990), Schummer (1999), Serizawa (1999) and Moreno and Moscoso (2011).

[3]A rule is inverse-dictatorial if there exists some agent who always gets nothing. In a two-agent economy, it is equivalent to a dicatorial rule.

## 1.2 The Model

Throughout this paper, for all $x, y \in \mathbb{R}^m$ where $m \in \mathbb{N}$, $x \geq y$ means that $x_k \geq y_k$, $\forall k = 1, ..., m$; $x > y$ means that $x_k > y_k$, $\forall k = 1, ..., m$. The latter will be the order that we refer to when we consider totally ordered sets in $\mathbb{R}^m$. Let $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x \geq 0\}$, $\mathring{\mathbb{R}}^m_+ = \{x \in \mathbb{R}^m | x > 0\}$, and $\partial\mathbb{R}^m_+ = \mathbb{R}^m_+ \backslash \mathring{\mathbb{R}}^m_+$. For any subsets $S_1$ and $S_2$ of $\mathbb{R}^m_+$, $S_1 + S_2 = \{s_1 + s_2 | s_1 \in S_1,\ s_2 \in S_2\}$, and similarly $S_1 - S_2 = \{s_1 - s_2 | s_1 \in S_1,\ s_2 \in S_2\}$.

Fix the set of perfectly divisible goods $L = \{1, ..., l\}$, $l \in \mathbb{N}$. Let $\mathbb{R}^l_+$ be the commodity space. Up to Section 3, every agent is assumed to have a standard Leontief preference on $\mathbb{R}^l_+$, which can be represented by a utility function $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$, $\forall x \in \mathbb{R}^l_+$, where $x^k$ denotes the amount of the k-th good, $\lambda_k > 0$, $\forall k \in L$, and $\sum_{k \in L} \lambda_k = 1$ for normalization. Let $\mathcal{U}$ denote the set of all such utility functions[4]. We will generalize this preference domain in Section 4.

**Definition 1.** *Let $u \in \mathcal{U}$ with $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$ be given. We call $\gamma = \{(\lambda_1 t, ..., \lambda_l t) \in \mathbb{R}^l_+ | t \in \mathbb{R}_+\}$ the critical set of the preference $u$.*

A critical set of a preference $u \in \mathcal{U}$ consists of all the minimum commodity bundles required to achieve given utility levels. It is a ray starting from the origin, and thus a connected, totally ordered and closed subset in $\mathbb{R}^l_+$. It is easy to see that $\gamma$ is uniquely defined for each $u \in \mathcal{U}$. Hence, in the following, we will interchangeably use $u$ and $\gamma$ as needed.

An economy $E$ is a triple $(N, u_N, \omega)$ where $N \subseteq \mathbb{N}$ is a nonempty finite set of agents, $u_N = (u_i)_{i \in N}$ with $u_i \in \mathcal{U}$, $\forall i \in N$, is a preference profile, and $\omega \in \mathbb{R}^l_+$ is the social endowment of the economy. Up to Section 7, the resources are assumed to be collectively

[4]We normalize the utility functions so that our rules only care about the ordinal properties. However, it is not necessary for our result. It can be easily shown that any rule satisfying efficiency, strategy-proofness and consistency only takes into account the ordinal properties.

owned. In Section 8, we consider the case where every agent has a private endowment and their endowments are put together to be divided. Let $\mathcal{E}$ denote the set of all economies.

Given $(N, \omega)$, the set of all feasible allocations is usually defined as $A(N, \omega) = \{x \in \mathbb{R}_+^{|N| \times l} | \sum_{i \in N} x_i \leq \omega\}$, where $x_i$ is the $l$ dimensional bundle for agent $i$. We further require that the bundle of each agent is in his critical set. The reason is that the Leontief preferences are not strictly monotone, so society would like to keep the redundant goods in this economy for alternative use, in the spirit of non-wastefulness. Note that our main result does not hold when the allocations are allowed to be wasteful. A counter-example will be given at the end of Section 3.

Formally, for any economy $E = (N, u_N, \omega)$, we consider the restriction of $A(N, \omega)$ on the critical sets, $A^*(E) = A(N, \omega) \cap \prod_{i \in N} \gamma_i$ where $\gamma_i$ is the critical set of $u_i$. Let $\mathcal{A}^* = \{A^*(E) | E \in \mathcal{E}\}$.

**Definition 2.** *An allocation rule (or rule for simplicity) is a mapping $\mu : \mathcal{E} \to \mathcal{A}^*$ with $\mu(E) \in A^*(E)$, assigning to each economy a non-wasteful feasible allocation. For any $i \in N$, $\mu_i(E)$ denotes the bundle allocated to agent $i$.*

For notational simplicity, we write $\mu(u_N)$ (or $\mu(\omega)$) to denote $\mu(N, u_N, \omega)$, when $(N, \omega)$ (or $(N, u_N)$) is fixed.

Our normative requirements on rules are all very familiar in the literature (see the Introduction).

**(I) Efficiency**

Efficiency naturally requires that a rule always assigns Pareto optimal allocations.

Given $E = (N, u_N, \omega)$, an allocation $x \in A(N, \omega)$ is *efficient* if there exists no $y \in A(N, \omega)$ such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$, and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. A rule $\mu$ is *efficient* (EFFN) if $\mu(E)$ is efficient for every $E \in \mathcal{E}$.

**Lemma 1.** *Given $E = (N, u_N, \omega)$, an allocation $x \in A^*(E)$ is efficient if and only if $\sum_{i \in N} x_i^k = \omega^k$ for some $k \in L$, where $x_i^k$ denotes the amount of good $k$ given to agent $i$.*

*Proof.* For sufficiency, suppose the contrary that there exists $y \in A(N, \omega)$ such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$, and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. Then $y_i \geq x_i$ for all $i \in N$ and $y_j > x_j$ for some $j \in N$, since $x_i \in \gamma_i$, $\forall i \in N$. Hence, $\sum_{i \in N} y_i > \sum_{i \in N} x_i$, and thus $\sum_{i \in N} y_i^k > \sum_{i \in N} x_i^k = \omega^k$, which contradicts feasibility. For necessity, suppose the contrary that $\sum_{i \in N} x_i < \omega$. Then consider the allocation $y \in A(N, \omega)$ such that $y_i = x_i$, $\forall i \in N \backslash \{j\}$, and $y_j = x_j + \omega - \sum_{i \in N} x_i > x_j$. Clearly, it implies that $x$ is not efficient, which is a contradiction. □

**(II) Incentive compatibility**

We require the familiar strategy-proofness and its strengthening as group strategy-proofness.

Let $\mathcal{U}_S = \mathcal{U}^{|S|}$, $\forall S \subseteq N$, and $\mathcal{U}_N$ is the set of all preference profiles. For any $S \subseteq N$, we denote by $(u'_S, u_{-S})$ the vector $u_N \in U_N$ with $u_i$ replaced by $u'_i$, $\forall i \in S$. If $S = \{i\}$, we simply write $(u'_i, u_{-i})$.

A rule $\mu$ is *strategy-proof* (SP) if $\forall (N, u_N, \omega)$, $\forall i \in N$, $\forall u'_i \in \mathcal{U}$, $u_i(\mu_i(u_N)) \geq u_i(\mu_i(u'_i, u_{-i}))$.

A rule $\mu$ is *group strategy-proof* (GSP) if $\forall (N, u_N, \omega)$, there does not exist $S \subseteq N$ and $u'_S \in \mathcal{U}_S$ such that $u_i(\mu_i(u_N)) \leq u_i(\mu_i(u'_S, u_{-S}))$, $\forall i \in S$, and at least one inequality is strict.

**(III) Fairness**

There are four classic fairness axioms: anonymity, envy-freeness, resource monotonicity and population monotonicity. Envy-freeness and resource monotonicity are known to be very demanding and usually incompatible.

Let $\pi$ be a bijection on $\mathbb{N}$. A rule $\mu$ is *anonymous* (ANON) if $\forall \pi$, $\forall (N, u_N, \omega)$, $\forall i \in N$, $\mu_i(N, u_N, \omega) = \mu_{\pi(i)}(\pi(N), (u_{\pi(j)})_{\pi(N)}, \omega)$ where $u_{\pi(j)} = u_j$, $\forall j \in N$.

**Remark 1.** *If $\mu$ is ANON, then for any $(N, u_N, \omega)$ such that $u_i = u_j$, $i, j \in N$, $\mu_i(N, u_N, \omega) = \mu_j(N, u_N, \omega)$.*

A rule $\mu$ is *envy-free* (EF) if $\forall(N, u_N, \omega)$, $\forall i, j \in N$, $u_i(\mu_i(N, u_N, \omega)) \geq u_i(\mu_j(N, u_N, \omega))$.

A rule $\mu$ is *resource monotonic* (RM) if $\forall(N, u_N)$, $\forall \omega, \omega' \in \mathbb{R}^l_+$, $\omega > \omega'$ implies that $u_i(\mu_i(\omega)) > u_i(\mu_i(\omega'))$, $\forall i \in N$.

There is another version of resource monotonicity. It states that $\forall(N, u_N)$, $\forall \omega, \omega' \in \mathbb{R}^l_+$, $\omega \geq \omega'$ implies that $u_i(\mu_i(\omega)) \geq u_i(\mu_i(\omega'))$, $\forall i \in N$. In general, these two versions do not imply each other. However, our rules below satisfy both of them, and the first one combined with the other axioms implies the second by our characterization result.

A rule $\mu$ is *population monotonic* (PM) if $\forall(N, u_N, \omega)$, $\forall N' \subseteq N$ and $N' \neq \emptyset$, $\forall i \in N'$, $u_i(\mu_i(N', u_{N'}, \omega)) \geq u_i(\mu_i(N, u_N, \omega))$.

**(IV) Consistency**

Consistency has played an important role in the rationing literature and also in the fair division problems of discrete goods.

A rule $\mu$ is *consistent* (CST) if $\forall(N, u_N, \omega)$, $\forall N' \subseteq N$ and $N' \neq \emptyset$, $\forall i \in N'$, $\mu_i(N, u_N, \omega) = \mu_i(N', u_{N'}, \omega - \sum_{j \in N \setminus N'} \mu_j(N, u_N, \omega))$.

Note that to check consistency, it is equivalent to check whether the corresponding condition holds when $|N'| = |N| - 1$.

**Remark 2.** *It is easy to see that if a rule is consistent and resource monotonic (no matter which version of resource monotonicity is adopted), then it must be population monotonic. In the following, if a rule is CST and RM, we will just keep in mind that it is also PM without even mentioning in the theorems.*

## 1.3 Generalized Egalitarian Rules

Let $f : \mathbb{D} \to \mathbb{R}^n$ where $\mathbb{D} \subseteq \mathbb{R}^m$ and $m, n \in \mathbb{N}$ be an arbitrary function. We say $f$ is strictly increasing if $\forall x, y \in \mathbb{R}^m,\ x > y$ implies that $f(x) > f(y)$.

Suppose that $W : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ is a strictly increasing and continuous function. Given an economy $E = (N, u_N, \omega)$, let $A^W(E) = \{x \in A^*(E) | W(x_i) = W(x_j), \forall i, j \in N\}$.

**Lemma 2.** *$A^W(E)$ is a totally ordered and closed set in $\mathbb{R}^{|N| \times l}$. In particular,* $\max A^W(E)$ *exists.*

*Proof.* To show that $A^W(E)$ is totally ordered, let $x, y \in A^W(E)$ such that $x \neq y$ be given. Suppose without loss of generality (WLOG) that $x_j < y_j$ for some $j \in N$. By the definition of $A^W(E)$ and the properties of $W$, we know that $\forall i \in N,\ x_i, y_i \in \gamma_i$, and $W(x_i) = W(x_j) < W(y_j) = W(y_i)$. Since the $\gamma_i$'s are totally ordered sets and $W$ is strictly increasing, then $x_i < y_i,\ \forall i \in N$, and thus $x < y$.

To see that $\max A^W(E)$ exists, note that $A^*(E)$ is closed and $W$ is continuous. Moreover, $A^W(E)$ is nonempty and bounded. Thus, $\max A^W(E)$ exists. □

Lemma 2 guarantees that the following rule is well-defined.

**Definition 3.** *A rule $\mu$ is called a generalized egalitarian rule, if there is a strictly increasing continuous function $W : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ such that for all $E \in \mathcal{E}$, $\mu(E) = \max A^W(E)$.*

*Let $\mathcal{M}$ denote the class of generalized egalitarian rules. We write $\mu^W$ when we want to indicate that $\mu$ is generated by $W$.*

We give two interpretations of our rules. One is in terms of a benchmark preference on the commodity space. The other is related to "equal opportunity allocations".

First, suppose that society has a benchmark preference over the commodity space which is represented by $W$.[5] Then $\mu^W$ assigns to each agent the same welfare level according to this benchmark preference of society. We use two examples to explain.

**Example 1** : *Equalizing total wealth.*

Fix a price vector $p \in \mathring{\mathbb{R}}^l_+$. Let $W(x) = p \cdot x$, $\forall x \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$. In this case, society wants the agents to get the same total wealth. The indifference classes of the benchmark preference are just the budget lines. See Figure 1 for an illustration in a two-good economy.

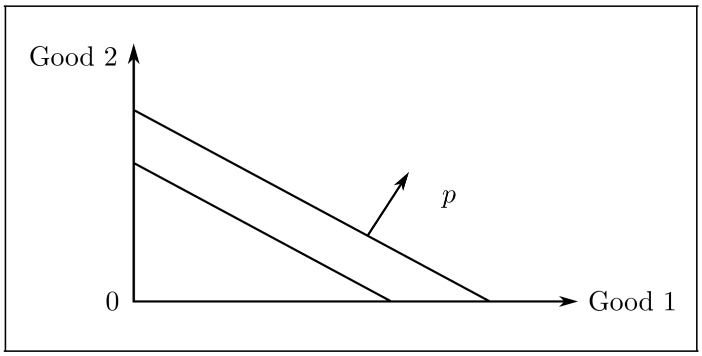


Figure 1.1 : Equalizing total wealth

**Example 2** : *Egalitarian – equivalent (EE) rules.*

The spirit of the classic EE rule is that every agent should get "equal" share of the social endowment. The difficulty is to find a way of measuring these shares in a world of ordinal preferences (Moulin, 1995). Pazner and Schmeidler (1978) were the first to propose a solution. It assigns an allocation at which the agents are indifferent between their bundles and the same fraction of the social endowment. In our context, that is, $\mu(E) = \max\{x \in A^*(E) | u_i(x_i) = u_i(t\omega), \forall i \in N, t \in \mathbb{R}_+\}$. However, the classic EE rule is not resource monotonic. Then the $e$-EE rule is proposed to overcome this drawback. The $e$-EE rule fixes

[5]The value of $W$ on $\partial\mathbb{R}^l_+ \backslash \{\mathbf{0}\}$ is irrelevant, since $A^*(E) \cap \partial\mathbb{R}^l_+ = \{\mathbf{0}\}$. More rigorously, $W$ represents a benchmark preference on the interior and the origin of the commpodity space.

an arbitrary reference bundle $e \in \mathring{\mathbb{R}}^l_+$, and gives the agents the shares between which and $te$ they feel indifferent, where $t$ is taken as high as possible. Even more generally, fix a strictly increasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}^l_+$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty, \forall k \in L$, and $\varphi(0) = \mathbf{0}$. We can define the $\varphi$-EE rule by $\mu(E) = \max\{x \in A^*(E) | u_i(x_i) = u_i(\varphi(t)), \forall i \in N, t \in \mathbb{R}_+\}$. The $\varphi$-EE rule makes all agents indifferent between their shares and the same commodity bundle on the reference curve, i.e., $\varphi(t^*)$ for some $t^* \in \mathbb{R}_+$. Hence, these shares are "equal" as viewed by society. Note that the $e$-EE rule is the $\varphi$-EE rule with $\varphi(t) = te$.

We check that on the domain of Leontief preferences, the $\varphi$-EE rule is a special case of the generalized egalitarian rules. Note that when $\omega$ is fixed, the classic EE rule with $\varphi(t) = t\omega$ is also a special case.

**Lemma 3.** *Let $\mu$ be a $\varphi$-EE rule. Define for all $y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$, $W(y) = t$ if and only if $y \in \{\varphi(t)\} - \partial\mathbb{R}^l_+$. Then $\mu = \mu^W$.*

*Proof.* First, since $\varphi$ is continuous and strictly increasing, $W$ is well-defined, and moreover, continuous and strictly increasing.

Next, fix $E = (N, u_N, \omega)$. Observe that $\forall x \in A^*(E)$ and $\forall i \in N$, $W(x_i) = t$ if and only if $x_i \in \{\varphi(t)\} - \partial\mathbb{R}^l_+$, which is equivalent to $u_i(x_i) = u_i(\varphi(t))$ since $x_i \in \gamma_i$. Hence, $\mu = \mu^W$ by the definitions. □

Figure 2 shows in a two-commodity space the indifference classes of the benchmark preference $W$ which is defined from $\varphi$.

The second interpretation relates to "equal opportunity allocations" proposed by Thomson (1994). Such an allocation is obtained by having each agent choose by himself in *a common choice set*. In this way, it gives the agents equal opportunities. We reformulate the definition in our context.

Let $\mathcal{C}$ be a family of choice sets, where each $C \in \mathcal{C}$ is a nonempty subset of $\mathbb{R}^l_+$.

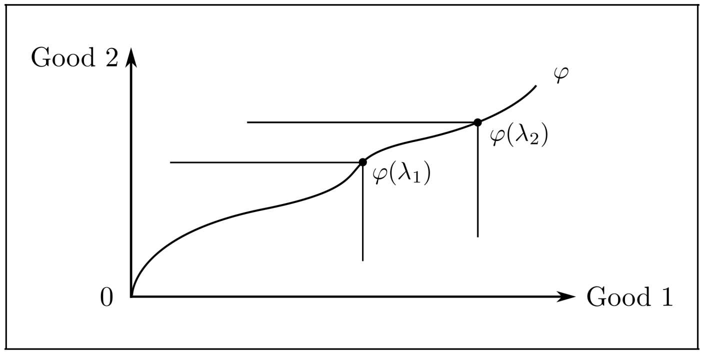


Figure 1.2 : $\varphi$-EE rules

**Definition 4.** (Thomson, 1994) *Given an economy $E = (N, u_N, \omega)$, a feasible allocation $x$ is an equal opportunity allocation relative to the family $\mathcal{C}$ if there exists $C \in \mathcal{C}$ such that $\forall i \in N$, $x_i \in \arg\max_{y \in C} u_i(y)$.*

**Lemma 4.** *Let $\mu^W$ be given. Suppose $C(t) = \{y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} | W(y) \leq t\}$ where $t \in \mathbb{R}_+$. Let $\mathcal{C} = \{C(t) | t \in \mathbb{R}_+\}$. Then $\mu^W(E) = \max\{x \in A^*(E) | x$ is an equal opportunity allocation relative to $\mathcal{C}\}$ for all $E \in \mathcal{E}$.*

*Proof.* Let $E$ be given. We only need to show that if $x \in A^*(E)$, then $W(x_i) = W(x_j), \forall i, j \in N$ is equivalent to that $x$ is an equal opportunity allocation relative to the family $\mathcal{C}$. If $W(x_i) = W(x_j), \forall i, j \in N$, then let $t = W(x_i)$, and thus $x_i$ is the optimal bundle in $C(t)$ for all $i$ since both $u_i$ and $W$ are strictly increasing. Conversely, suppose that $x_i$ is the optimal bundle in $C(t)$ for all $i$. If WLOG there exist $x_1$ and $x_2$, such that $W(x_1) > W(x_2)$, then we must have $t \geq W(x_1) > W(x_2)$. Thus there must exist $x_2' \in \gamma_2$ such that $x_2' > x_2$ and $W(x_2') < t$. It contradicts that $x_2$ is the optimal bundle in $C(t)$. □

Hence, a generalized egalitarian rule always picks the Pareto optimal equal opportunity allocation relative to the family of nested choice sets generated by $W$. In example 1, $\mathcal{C}$ is the class of all budget sets with a fixed price. In example 2, $\mathcal{C}$ is the class of box-shaped sets $C$ with $C = \{y | y \leq \varphi(\lambda)\}$.

Our first main result is a characterization of generalized egalitarian rules.

**Theorem 1.** *(i) If a rule $\mu$ is in $\mathcal{M}$, then it is efficient, resource monotonic, consistent, group strategy-proof, anonymous and envy-free.*

*(ii) Let a rule $\mu$ be efficient, resource monotonic and consistent. If $\mu$ is either strategy-proof and anonymous, or envy-free, then $\mu \in \mathcal{M}$.*

In fact, Theorem 1 also holds for a much larger preference domain which is the object of the next section.

The requirement of non-wasteful allocation is very important for Theorem 1. Consider a natural extension of our rules to those which divide up every good. That is, first apply a generalized egalitarian rule $\mu^W$ and then allocate the remaining goods equally among the agents. More precisely, this extended rule $\bar{\mu}$ assigns for all $E = (N, \mu_N, \omega)$ and for all $i \in N$, $\bar{\mu}_i(E) = \mu_i^W(E) + \frac{1}{|N|}(\omega - \sum_{i \in N} \mu_i^W(E))$. We show that $\bar{\mu}$ is not SP by a counter-example. For simplicity suppose that $W = p \cdot x$ where $p > \mathbf{0}$. Let $E = (\{1, 2\}, (u_1, u_2), \omega)$ where (i) $\omega \in \mathbb{R}^2_+$ and $\omega_2$ is large enough so that good 2 is always available in the following discussion; (ii) the slope of the critical set of $u_1$ is greater than that of $u_2$. Let $u_1'$ be such that the slope of its critical set is in between those of $u_1$ and $u_2$. See Figure 3 for an illustration. Let $E'$ be $E$ with $u_1$ replaced by $u_1'$. Suppose that $\mu^W(E) = (x_1, x_2)$, $\mu^W(E') = (y_1, y_2)$, $\bar{\mu}(E) = (\bar{x}_1, \bar{x}_2)$ and $\bar{\mu}(E') = (\bar{y}_1, \bar{y}_2)$. Since $\omega_2$ is large enough, then it is always good 1 that is divided up. We check that $y_1^1 > x_1^1$. If $y_1^1 \leq x_1^1$, then $W(y_2) = W(y_1) < W(x_1) = W(x_2)$. Hence, $y_2 < x_2$, and thus $y_1^1 + y_2^1 < x_1^1 + x_2^1 = \omega_1$, which violates the efficiency of $\mu^W$. Once again let $\omega_2$ be large enough such that $\bar{y}_1^2 = y_1^2 + \frac{1}{2}(\omega_2 - y_1^2 - y_2^2) > x_1^2$. Then after dividing the remaining good 2, $\bar{y}_1 > x_1$, and thus $u_1(\bar{y}_1) > u_1(x_1) = u_1(\bar{x}_1)$. This example can be easily extended to economies with more goods.

Hence, if one wants a rule to allocate all the goods and be EFFN and SP, then one

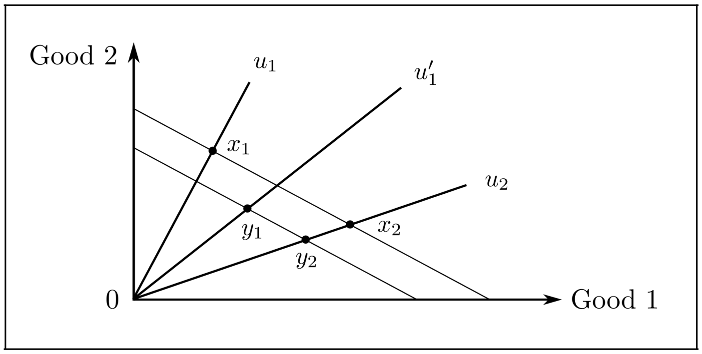


Figure 1.3 : A counter-example for wasteful allocation

must carefully design the way that the useless goods are divided. Nicolò (2004) provides such a rule in a two-agent two-good economy. However, there is no result yet in a general economy.

**Remark 3.** *In the characterization of Nicolò (2004), he introduces an incentive compatibility axiom stronger than strategy-proofness — fully implementability in truthful strategies. It requires that a rule is strategy-proof and moreover when a misreport of an agent does not change his own utility, the whole allocation is unaffected. Our rules satisfy this axiom if and only if* $\forall x, y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$, $x \geq y$ *and* $x \neq y$ *imply that* $W(x) > W(y)$.

## 1.4 Generalized Leontief Preferences

All the proofs of the results in this section are in the Appendix.

Let $\succsim$ be a complete and transitive binary relation on $\mathbb{R}^l_+$, $\succ$ and $\sim$ be the corresponding strict and indifferent relations. For all $x \in \mathbb{R}^l_+$, denote by $U_{\succsim}(x) = \{y \in \mathbb{R}^l_+ | y \succsim x\}$ the upper contour set of $x$, and $I_{\succsim}(x) = \{y \in \mathbb{R}^l_+ | y \sim x\}$ its indifference class.

**Definition 5.** *The set of generalized Leontief preferences is defined by* $\mathcal{D} = \{ \succsim$ *on* $\mathbb{R}^l_+ |$ $\succsim$

*is continuous and locally non-satiated, and* $\forall x \in \mathbb{R}^l_+$, $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ *for some* $a \in \mathbb{R}^l_+\}$.

**Lemma 5.** *If* $\succsim\ \in \mathcal{D}$, *then*

*(i)* $\succsim$ *is monotone, i.e.,* $\forall x, y \in \mathbb{R}^l_+$, $x > y$ *implies that* $x \succ y$;

*(ii) for any* $x \in \mathbb{R}^l_+$, $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ *implies that* $I_{\succsim}(x) = \{a\} + \partial\mathbb{R}^l_+$.

**Definition 6.** *For any* $\succsim\ \in \mathcal{D}$, *define* $\gamma_{\succsim} = \{a \in \mathbb{R}^l_+ : U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ *for some* $x \in \mathbb{R}^l_+\}$ *to be the critical set of the preference* $\succsim$.

Clearly, Definition 6 generalizes Definition 1 on the domain of generalized Leontief preferences.

**Lemma 6.** *For any* $\succsim\ \in \mathcal{D}$,

*(i)* $\boldsymbol{0} \in \gamma_{\succsim}$, *and* $\gamma_{\succsim}$ *is unbounded;*

*(ii) if* $a, b \in \gamma_{\succsim}$ *and* $a \neq b$, *then either* $a < b$ *or* $a > b$, *i.e.,* $\gamma_{\succsim}$ *is totally ordered;*

*(iii)* $\gamma_{\succsim}$ *is connected;*

*(iv)* $\gamma_{\succsim}$ *is closed.*

Figure 4 shows the typical upper contour set, the indifference class and the critical set of a generalized Leontief preference in a two-good economy.

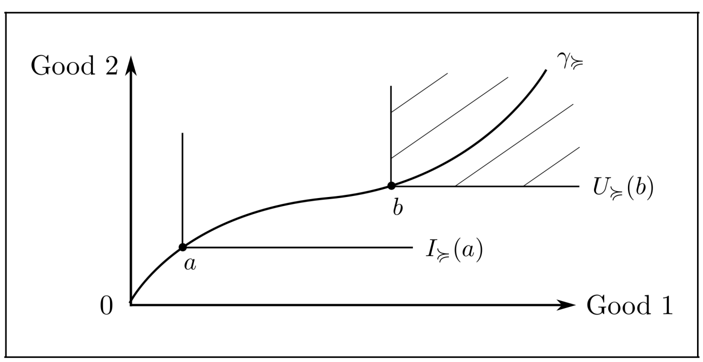


Figure 1.4 : A generalized Leontief preference in a two-good economy

**Proposition 1.** *For any* $\succsim \in \mathcal{D}$*,* $\succsim$ *is represented by* $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$*, where* $\zeta : \mathbb{R}_+ \to \gamma_{\succsim}$ *is a strictly increasing homeomorphism such that* $\sum_{k \in L} \zeta^k(t) = t$, $\forall t \in \mathbb{R}_+$.

For any $x \in \gamma_{\succsim}$, $x = \zeta(t)$ for some $t$, and thus $u(x) = t = \sum_{k \in L} x^k$. Hence, $u$ restricted on $\gamma_{\succsim}$ is a strictly increasing continuous function.

Let $\tilde{\mathcal{U}}$ be the set of all utility functions representing generalized Leontief preferences in the way specified in Proposition 1. Note that $\tilde{\mathcal{U}}$ is a generalization of $\mathcal{U}$, since for any standard Leontief preference represented by $u \in \mathcal{U}$ with $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$, $\zeta(t) = (\lambda_1 t, ..., \lambda_l t)$, $\forall t \in \mathbb{R}_+$, and thus $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, as well.

It is easy to see that under the larger preference domain $\tilde{\mathcal{U}}$, all the previous notions such as economy, rule and generalized egalitarian rule are still well-defined. Moreover, as we mentioned before, Theorem 1 still holds when $\mathcal{U}$ is replaced by $\tilde{\mathcal{U}}$.

Let $\tilde{\mathcal{M}}$ denote the class of generalized egalitarian rules under the domain $\tilde{\mathcal{U}}$. For simplicity, we will still use notations such as $E$, $A^*(E)$ and $\mu$ to denote the corresponding notions under the generalized preference domain.

**Theorem 2.** *(i) If a rule* $\mu$ *is in* $\tilde{\mathcal{M}}$*, then it is efficient, resource monotonic, consistent, group strategy-proof, anonymous and envy-free.*

*(ii) Let a rule* $\mu$ *be efficient, resource monotonic and consistent. If* $\mu$ *is either strategy-proof and anonymous, or envy-free, then* $\mu \in \tilde{\mathcal{M}}$*.*

## 1.5 The Proofs

Generally speaking, the structure of our problem has some resemblance to the "fixed path" methods in the rationing literature, such as the parametric method in Young (1987), and the fixed path rationing method in Moulin (1999). The essential idea of the proof is to investigate how the given axioms impact the range of the rules. We find that the range can be identified with some features which enable us to construct a benchmark preference.

Here we prove Theorem 2. In fact, the result of every step in the following is true under both preference domains. The proofs under $\mathcal{U}$ just involve less cases to check. For the simplicity of presentation, we assume that a rule assigns to every agent an unbounded bundle when the endowment increases, i.e., $\forall(N, u_N)$, $\forall i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+)= \{x_i \in \mathbb{R}^l_+ | \mu_i(N, u_N, \omega) = x_i$ for some $\omega \in \mathbb{R}^l_+\}$ is an unbounded subset in $\mathbb{R}^l_+$. This assumption is not necessary. The relaxation of it will be discussed in the Appendix.

**Step 1.** If $\mu$ is EFFN and RM, then

(i) $\forall(N, u_N)$, $\forall x, x' \in \mu(N, u_N, \mathbb{R}^l_+)$ such that $x \neq x'$, either $x_i < x'_i$, $\forall i \in N$, or $x_i > x'_i$, $\forall i \in N$;

(ii) $\forall(N, u_N)$, $\forall i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+) = \gamma_i$.

*Proof.* Let $(N, u_N)$ be given. Suppose WLOG that $N = \{1, ..., n\}$.

(i) Assume that $\mu(\omega) = x$, $\mu(\omega') = x'$, $\omega, \omega' \in \mathbb{R}^l_+$, and $x \neq x'$.

First observe that if $x_j < x'_j$ for some $j \in N$, then $x_i \leq x'_i$ for all $i \in N$. Suppose the contrary WLOG that $x_1 < x'_1$ and $x_2 > x'_2$. Then $\sum_{i \in N} \min\{x_i, x'_i\} < \sum_{i \in N} x_i \leq \omega$, and $\sum_{i \in N} \min\{x_i, x'_i\} < \sum_{i \in N} x'_i \leq \omega'$. Since $\mu$ is RM, then $\mu_i(\sum_{i \in N} \min\{x_i, x'_i\}) < \min\{x_i, x'_i\}$, $\forall i \in N$, which violates the efficiency of $\mu$.

Next note that if $y \in \mu(\mathbb{R}^l_+)$, then $\mu(\sum_{i \in N} y_i) = y$. Suppose the contrary WLOG that $\mu(\sum_{i \in N} y_i) = y'$ and $y_1 < y'_1$. By our previous result, $y_i \leq y'_i$, $\forall i \in N$. Thus $\sum_{i \in N} y'_i > \sum_{i \in N} y_i$, which violates feasibility.

Hence, we can take $\omega = \sum_{i \in N} x_i$ and $\omega' = \sum_{i \in N} x'_i$. Suppose WLOG that $x_1 \neq x'_1$. If $x_1 < x'_1$, then we know that $x_i \leq x'_i$, $\forall i \in N$. Thus, $\omega < \omega'$. Since $\mu$ is RM, then $x_i < x'_i$, $\forall i \in N$. Similarly, if $x_1 > x'_1$, then $x_i > x'_i$, $\forall i \in N$.

(ii) Suppose the contrary WLOG that $a \in \gamma_1 \setminus \mu_1(\mathbb{R}^l_+)$. Since $\mathbf{0} \in \mu_1(\mathbb{R}^l_+)$ and $\mu_1(\mathbb{R}^l_+)$ is unbounded, then $\underline{\nu} = \{x \in \mu(\mathbb{R}^l_+) | x_1 < a\}$ and $\bar{\nu} = \{x \in \mu(\mathbb{R}^l_+) | x_1 > a\}$ are nonempty. Let $\underline{\omega} = \sup\{\sum_{i \in N} x_i | x \in \underline{\nu}\}$ [6] and $\bar{\omega} = \inf\{\sum_{i \in N} x_i | x \in \bar{\nu}\}$. By (i), $\underline{\nu} \cup \bar{\nu} = \mu(\mathbb{R}^l_+)$ is totally ordered, so $\underline{\omega}$ and $\bar{\omega}$ are well-defined, and $\underline{\omega} \leq \bar{\omega}$. If $\underline{\omega} < \bar{\omega}$, then pick $\omega$ such that $\underline{\omega} < \omega < \bar{\omega}$. By the choice of $\omega$, $\mu(\omega) \notin \underline{\nu} \cup \bar{\nu}$, which is a contradiction. If $\underline{\omega} = \bar{\omega}$, let $y = \sup \underline{\nu} = \inf \bar{\nu}$, and then $y_1 = a$. Let $(y'_i)_{i \in N} = \mu(\sum_{i \in N} y_i)$. By assumption $y'_1 \neq y_1$. If $y_1 < y'_1$, then $y' \in \bar{\nu}$ and thus $y'_i \geq y_i$, $\forall i \in N$. Hence, $\sum_{i \in N} y'_i > \sum_{i \in N} y_i$, which violates the feasibility. If $y_1 > y'_1$, then by a similar argument the efficiency is violated. □

**Step 2.** If $\mu \in \tilde{\mathcal{M}}$ is EFFN, RM and CST, then

(i) $\forall (N, u_N)$, $\forall N' \subseteq N$, $(x_i)_{i \in N} \in \mu(N, u_N, \mathbb{R}^l_+)$ implies that $(x_i)_{i \in N'} \in \mu(N', u_{N'}, \mathbb{R}^l_+)$;

(ii) $\forall (N_1, u_{N_1})$ and $(N_2, u_{N_2})$ such that $N_1 \cap N_2 = \emptyset$, $\forall (x_i)_{i \in N_1} \in \mu(N_1, u_{N_1}, \mathbb{R}^l_+)$ and $(x_i)_{i \in N_2} \in \mu(N_2, u_{N_2}, \mathbb{R}^l_+)$, if for some $i_1 \in N_1$ and $i_2 \in N_2$, $(x_{i_1}, x_{i_2}) \in \mu(\{i_1, i_2\}, (u_{i_1}, u_{i_2}), \mathbb{R}^l_+)$, then $(x_i)_{i \in N} \in \mu(N, u_N, \mathbb{R}^l_+)$ where $N = N_1 \cup N_2$ and $u_N = (u_{N_1}, u_{N_2})$.

*Proof.* Obviously, (i) follows from Step 1 (i) and the definition of consistency.

For (ii), suppose the contrary that under the required condition, $(x_i)_{i \in N} \notin \mu(N, u_N, \mathbb{R}^l_+)$. Then assume that $\mu(N, u_N, \sum_{i \in N} x_i) = (x'_i)_{i \in N} \neq (x_i)_{i \in N}$. Thus, there must exist some $j \in N$ such that $x'_j < x_j$. Suppose WLOG that $j \in N_1$. By (i), we know that $(x'_i)_{i \in N_1} \in \mu(N_1, u_{N_1}, \mathbb{R}^l_+)$, $(x'_{i_1}, x'_{i_2}) \in \mu(\{i_1, i_2\}, (u_{i_1}, u_{i_2}), \mathbb{R}^l_+)$, and $(x'_i)_{i \in N_2} \in \mu(N_2, u_{N_2}, \mathbb{R}^l_+)$. From our assumption and Step 1, we have that $x'_i < x_i$, $\forall i \in N_1$, and thus $x'_{i_2} < x_{i_2}$, and finally $x'_i < x_i$, $\forall i \in N_2$. Hence, $\sum_{i \in N} x'_i < \sum_{i \in N} x_i$, which violates that $\mu$ is EFFN. □

**Remark 4.** *It can also be shown that if $\mu$ is EFFN and RM, then both (i) and (ii) of Step 2 are sufficient conditions for $\mu$ to be CST.*

[6] For all $A \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, $(\sup A)_k = \sup\{a_k : a \in A\}$, $k = 1, ..., m$; $\inf A$ is similarly defined.

**Step 3.** Suppose that $\mu$ is EFFN, RM and CST. Then $\mu$ is SP if and only if $\forall(N, u_N)$ such that $|N| = 2$, $\forall i \in N$, $\forall u'_i \in \tilde{\mathcal{U}}$, if $(x_i, x_{-i}) \in \mu(N, u_N, \mathbb{R}^l_+)$ and $(x'_i, x_{-i}) \in \mu(N, u'_N, \mathbb{R}^l_+)$ where $u'_N = (u'_i, u_{-i})$, then $x_i \nless x'_i$.

*Proof.* For necessity, suppose the contrary WLOG that $N = \{1, 2\}$, and under the required condition, $x_1 < x'_1$. Since $\gamma'_1$ is connected, we can find $y'_1 \in \gamma'_1$ such that $y'_1 \in \{x_1\} + \partial\mathbb{R}^l_+$. See Figure 5 for an illustration in a two-good economy.

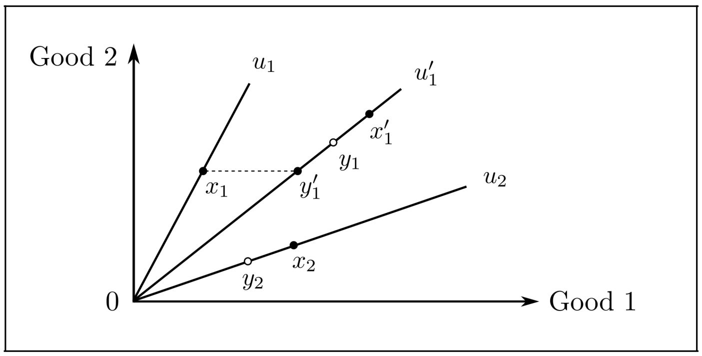


Figure 1.5 : Necessity for Step 3 (strategy-proofness)

Let $\omega = y'_1 + x_2$. By Step 1 (i), $\mu(N, u_N, \omega) = (x_1, x_2)$. We assume that $\mu(N, u'_N, \omega) = (y_1, y_2)$. Since $(x'_1, x_2) \in \mu(N, u'_N, \mathbb{R}^l_+)$ and $x'_1 + x_2 > x_1 + x_2$, then $y_1 < x'_1$ and $y_2 < x_2$. Thus by efficiency, $y_1 > y'_1 \geq x_1$. This means that in the economy $(N, u_N, \omega)$, agent 1 has incentive to misreport his preference, which violates that $\mu$ is SP.

For sufficiency, given the required assumption, we want to show that $\mu$ is SP. WLOG let $(N, u_N, \omega)$ where $N = \{1, ..., n\}$, and $u'_1 \in \tilde{\mathcal{U}}$ be given. Let $\mu(N, u_N, \omega) = (x_i)_{i \in N}$, and $\mu(N, u'_N, \omega) = (x'_i)_{i \in N}$ where $u'_N = (u'_1, u_{-1})$. See Figure 6.

We can find $y_1 \in \gamma'_1$ such that $(y_1, x_2) \in \mu(\{1, 2\}, (u'_1, u_2), \mathbb{R}^l_+)$. By consistency, $(x_1, x_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. By the required assumption, $x_1 \nless y_1$. Hence, if $x'_1 \leq y_1$, then $x_1 \nless x'_1$. Consider the other case that $x'_1 > y_1$. By consistency, $(x_2, ..., x_n) \in$

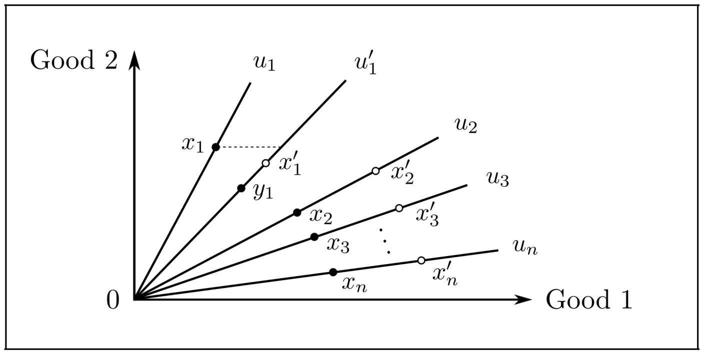


Figure 1.6 : Sufficiency for Step 3 (strategy-proofness)

$\mu(N\setminus\{1\}, u_{N\setminus\{1\}}, \mathbb{R}^l_+)$. From Step 2, consider $(N_1, u_{N_1}) = (\{1\}, u'_1)$, $(N_2, u_{N_2}) = (N\setminus\{1\}, u_{N\setminus\{1\}})$, and thus $(y_1, x_2, ..., x_n) \in \mu(N, u'_N, \mathbb{R}^l_+)$. Hence, $x'_i > x_i$, $\forall i = 2, ..., n$. If $x'_1 > x_1$, then $\omega \geq \sum_{i\in N} x'_i > \sum_{i\in N} x_i$, which violates the efficiency. Hence, $x_1 \not< x'_1$ and agent 1 has no incentive to misreport his preference. $\square$

**Step 4.** A rule $\mu \in \tilde{\mathcal{M}}$ if and only if $\mu$ is EFFN, RM, CST, SP and ANON.

*Proof.* For necessity, let $\mu \in \tilde{\mathcal{M}}$ and $(N, u_N, \omega)$ be given. To check efficiency, by Lemma 1, we only need to check that some commodity is divided up. Suppose the contrary that $\mu(N, u_N, \omega) = x$ and $\sum_{i\in N} x_i < \omega$. We can find for each $i \in N$ $x'_i \in \gamma_i$ such that $x'_i > x_i$ and $\sum_{i\in N} x'_i \leq \omega$, since $\gamma_i$'s are connected. Pick $t \in \mathbb{R}_+$ such that $W(x_i) < t < W(x'_i)$, $\forall i \in N$. Since $W$ is continuous and $\gamma_i$'s are connected, then $W(\gamma_i)$'s are connected. Thus for each $i \in N$ there exists $y_i \in \gamma_i$ such that $W(y_i) = t$. Clearly, $\sum_{i\in N} y_i < \sum_{i\in N} x'_i \leq \omega$, which contradicts that $\mu(N, u_N, \omega) = x$ by the definition of $\mu$.

To verify that $\mu$ is RM, fix $\omega'$ such that $\omega' > \omega$. Then use the similar argument as above, we can show that the bundle allocated to every agent is strictly increased.

Consistency follows from the definition of $\mu$, the efficiency of $\mu$, and the assumption that $W$ is strictly increasing.

Strategy-proofness follows from Step 3 and strict increasingness of $W$.

Lastly, anonymity is simply because $\mu$ does not depend on agents' names, but their preferences.

For sufficiency, suppose that $\mu$ is EFFN, RM, CST, SP and ANON. Fix $\bar{u} \in \tilde{\mathcal{U}}$. Define $W : \mathring{R}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ as follows. For any $x \in \mathring{R}^l_+ \cup \{\mathbf{0}\}$, choose $u_x \in \tilde{\mathcal{U}}$ such that its critical set $\gamma_x$ contains $x$. Choose $N = \{1, 2\}$, $u_1 = \bar{u}$, and $u_2 = u_x$. From Step 1, we know that there uniquely exists $\bar{x} \in \bar{\gamma}$ such that $(\bar{x}, x) \in \mu(N, u_N, \mathbb{R}^l_+)$. Define $W(x) = \bar{u}(\bar{x})$. The choice of $u_x$ does not matter, since for any other $u'_x \in \tilde{\mathcal{U}}$ such that $x \in \gamma'_x$ and the corresponding $\bar{x}' \neq \bar{x}$, WLOG say $\bar{x}' < \bar{x}$, then there must be an $x' \in \gamma_x$ such that $x' < x$ and $(\bar{x}', x') \in \mu(N, (\bar{u}, u_x), \mathbb{R}^l_+)$, which contradicts that $\mu$ is SP by Step 3. See Figure 7 for an illustration in a two-good economy. Hence, $W$ is well-defined. Note that for any $x \in \bar{\gamma}$, we can pick $u_x = \bar{u}$. Since $\mu$ is ANON, then $\mu_i(N, u_N, 2x) = x$, $i = 1, 2$, and thus $W(x) = \bar{u}(x)$ for all $x \in \bar{\gamma}$.

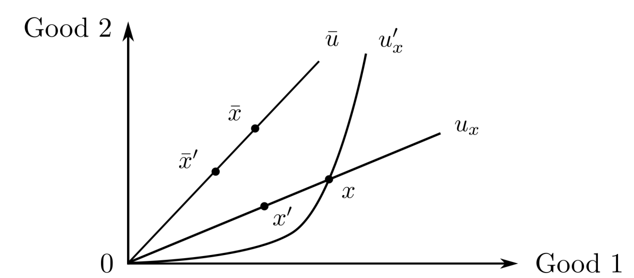


Figure 1.7 : Independence of the choice of $u_x$

To check that $W$ is strictly increasing, let $x, y \in \mathbb{R}^l_+$ such that $x < y$. We can find $u \in \tilde{\mathcal{U}}$ whose critical set contains both $x$ and $y$. Find $\bar{x}, \bar{y} \in \bar{\gamma}$ such that $(x, \bar{x}), (y, \bar{y}) \in \mu(\{1, 2\}, (\bar{u}, u), \mathbb{R}^l_+)$. Clearly, $\bar{x} < \bar{y}$, and thus $W(x) = \bar{u}(\bar{x}) < \bar{u}(\bar{y}) = W(y)$.

To verify that $W$ is continuous, we only need to check that $W^{-1}((t, \infty))$ and $W^{-1}([0, s))$

are open sets in $\mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$ when $t \geq 0$ and $s > 0$. Let $t \geq 0$ and $x \in W^{-1}((t,\infty))$ be given. Let $u_x$ and $\bar{x}$ be correspondingly given. By Proposition 1, we can find $\bar{x}_t \in \bar{\gamma}$ such that $\bar{u}(\bar{x}_t) = t$. By Step 1 (ii), there exists $x_t \in \gamma_x$ such that $W(x_t) = t$. See Figure 8. Since $x \in \gamma_x$ and $W(x) > t$, then $x > x_t$. Thus there exists $\epsilon > 0$ such that $B_\epsilon(x) = \{y \in \mathbb{R}^l_+ | \, \|y - x\| < \epsilon\} \subseteq \{x_t\} + \mathring{\mathbb{R}}^l_+$. For all $y \in B_\epsilon(x)$, $y > x_t$, and thus $W(y) > W(x_t) = t$. Hence, $B_\epsilon(x) \subseteq W^{-1}((t,\infty))$, which implies that $W^{-1}((t,\infty))$ is open. Similarly, we have that $W^{-1}([0,s))$ is open for all $s > 0$.

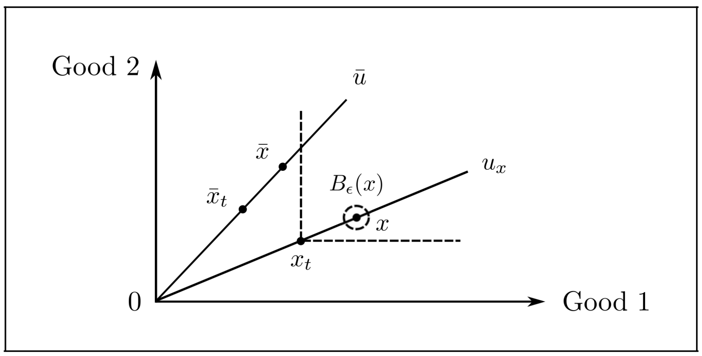


Figure 1.8 : The continuity of $W$

Finally, we check that $\forall E = (N, u_N, \omega)$, $\mu(E) = \max\{x \in A^*(E) | W(x_i) = W(x_j),\ \forall i, j \in N\}$. Suppose that $\mu(E) = (x_i^*)_{i \in N}$. Fix $i, j \in N$, and $i \neq j$. Assume WLOG that $1 \notin N$. By the construction of $W$ and the anonymity of $\mu$, there exists $\bar{x}$ such that $(\bar{x}, x_i^*) \in \mu(\{1, i\}, (\bar{u}, u_i), \mathbb{R}^l_+)$. Since $\mu$ is CST, $(x_i^*, x_j^*) \in \mu(\{i, j\}, (u_i, u_j), \mathbb{R}^l_+)$. Using Step 2 (ii), consider $N_1 = \{1\}$, $N_2 = \{i, j\}$, we get that $(\bar{x}, x_i^*, x_j^*) \in \mu(\{1, i, j\}, (\bar{u}, u_i, u_j), \mathbb{R}^l_+)$. By the consistency of $\mu$, $(\bar{x}, x_j^*) \in \mu(\{1, j\}, (\bar{u}, u_j), \mathbb{R}^l_+)$. Since $\mu$ is ANON, $W(x_i^*) = W(x_j^*)$. Since $\mu$ is EFFN, $(x_i^*)_{i \in N} = \max\{x \in A^*(E) | W(x_i) = W(x_j),\ \forall i, j \in N\}$. □

**Step 5.** If $\mu$ is in $\tilde{\mathcal{M}}$, then $\mu$ is GSP.

*Proof.* Let $(N, u_N, \omega)$, $S \subseteq N$, and $u'_N = (u'_S, u_{-S})$ where $u'_S \in \tilde{\mathcal{U}}_S$ be given. Assume that

$\mu(N, u_N, \omega) = x$ and $\mu(N, u'_N, \omega) = x'$. Suppose the contrary that $\forall i \in S$, $u_i(x'_i) \geq u_i(x_i)$, and $\exists j \in S$ such that $u_j(x'_j) > u_j(x_j)$. Hence, $\forall i \in S$, $x'_i \geq x_i$ and $x'_j > x_j$. Thus $W(x'_j) > W(x_j)$, which by the definition of $\mu$ implies that $\forall i \in N \setminus S$, $x'_i > x_i$. Therefore, $\sum_{i \in N} x_i < \sum_{i \in N} x'_i \leq \omega$, which contradicts the efficiency of $\mu$. $\square$

**Step 6.** A rule $\mu$ is in $\tilde{\mathcal{M}}$ if and only if $\mu$ is EFFN, RM, CST and EF.

*Proof.* For necessity, let $\mu \in \tilde{\mathcal{M}}$ be given. We only need to check that $\mu$ is EF. This simply follows from the definition of $\mu$ and the assumption that $W$ is strictly increasing.

For sufficiency, suppose that $\mu$ is EFFN, RM, CST and EF. First we show that $\mu$ is ANON. Let a bijection $\pi$ on $\mathbb{N}$, and an economy $E = (N, u_N, \omega)$ be given. Let $E' = (\pi(N), (u_{\pi(i)})_{\pi(N)}, \omega)$ where $u_i = u_{\pi(i)}$, $\forall i \in N$. Assume that $\mu(E) = x$ and $\mu(E') = x'$. Suppose the contrary WLOG that $1, 2 \in N$ and $x_1 < x'_{\pi(1)}$ and $x_2 > x'_{\pi(2)}$. We can find $x'_1 \in \gamma_1$ such that $(x'_1, x'_{\pi(2)}) \in \mu(\{1,2\}, (u_1, u_2), \mathbb{R}^l_+)$. Note that $x'_1 < x_1 < x'_{\pi(1)}$ since $x'_{\pi(2)} < x_2$. See Figure 9.

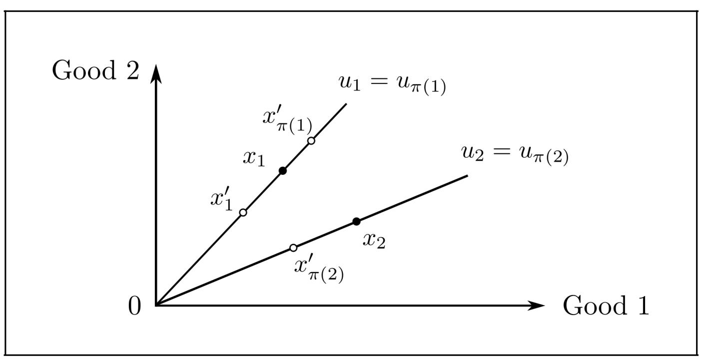


Figure 1.9 : The anonymity of $\mu$

Suppose that $\{1,2\} \cap \{\pi(1), \pi(2)\} = \emptyset$. Since $\mu$ is EFFN and EF, then $\mu(\{2, \pi(2)\}, (u_2, u_2), 2x'_{\pi(2)}) = (x'_{\pi(2)}, x'_{\pi(2)})$. Thus by Step 2 (ii), $(x'_1, x'_{\pi(2)}, x'_{\pi(2)}, x'_{\pi(1)}) \in$

$\mu(\{1,2,\pi(2),\pi(1)\},(u_1,u_2,u_2,u_1),\mathbb{R}^l_+)$, and agent 1 will envy agent $\pi(1)$ which is a contradiction. If $\{1,2\}\cap\{\pi(1),\pi(2)\}\neq\emptyset$, then pick $i_1,i_2\in\mathbb{N}$ such that $\{1,2,\pi(1),\pi(2)\}\cap\{i_1,i_2\}=\emptyset$. From the above result, we know that $(x'_{\pi(1)},x'_{\pi(2)})\in\mu(\{i_1,i_2\},(u_1,u_2),\mathbb{R}^l_+)$. Applying the same argument to the agents $1,2,i_1,i_2$ with the preferences $u_1,u_2,u_1,u_2$ respectively, we again will get a contradiction.

Now we only need to show that $\mu$ is SP. By Step 3, suppose WLOG that $\mu(\{1,2\},(u_1,u_2),\omega)=(x_1,x_2)$ and $\mu(\{1,2\},(u'_1,u_2),\omega')=(x'_1,x_2)$, and we want to check whether $x_1\nless x'_1$. Let $u_3=u'_1$. Since $\mu$ is ANON, then $\mu(\{3,2\},(u_3,u_2),\omega)=(x'_1,x_2)$. Since $\mu$ is CST, then by Step 2 $\mu(\{1,2,3\},(u_1,u_2,u_3),\omega'')=(x_1,x_2,x'_1)$ for some $\omega''\in\mathbb{R}^l_+$. Since $\mu$ is EF, then $u_1(x_1)\geq u_1(x'_1)$, and thus $x_1\nless x'_1$. $\square$

## 1.6 Tightness of the Characterization

By Theorem 2, a rule is in $\tilde{\mathcal{M}}$ if and only if one of the following equivalent conditions holds:

(i) it is EFFN, RM, CST, ANON and SP;

(ii) it is EFFN, RM, CST, ANON and GSP;

(iii) it is EFFN, RM, CST and EF.

Our characterization is tight with respect to all these axioms when there are at least two goods in the economy.[7] The tightness result for Theorem 1 is the same.

Drop the efficiency, and consider the rule $\bar{\mu}$ such that for all $E=(N,u_N,\omega)$, $\bar{\mu}(E)=\max\{x\in A^*(E)|W(x_i)=t,\forall i\in N;\sum_{i\in N}x_i\leq\omega-te\}$ where $W$ is as in Example 1, and $e$ is the unit vector in the commodity space. It can be checked that $\bar{\mu}$ is well-defined, and is RM, CST, ANON, GSP and EF. The key fact used to verify these properties is that if

[7] It is easy to see that if there is only one good in the economy, then efficiency and either anonymity or envy-freeness will suffice to characterize the rules tightly.

$W(\bar{\mu}_i(E)) = t$, $\forall i \in N$, then $\sum_{i\in N} \bar{\mu}_i^k(E) = \omega^k - t$ for some $k \in L$. However, the allocation given by this rule is never efficient when $\omega > \mathbf{0}$.

Drop the resource monotonicity, and the following rule $\bar{\mu}$ is EFFN, CST, ANON, GSP and EF. Here we define $\bar{\mu}$ in a two-good economy for simplicity, and it can be easily extended to the economies with more than two goods. Consider for each $t \in \mathbb{R}_+$, a parameterized indifference curve $q(t)$ such that: $q(t) = \{x \in \mathbb{R}^2_+ | x^1 + x^2 = t\}$ when $t \in [0, 2]$; $q(t) = \{x \in \mathbb{R}^2_+ | x^1 + (t-1)x^2 = t$, where $x^1 \geq 1$ or $(t-1)x^1 + x^2 = t$, where $x^1 \leq 1\}$ when $t \in [2, 4]$, and $q(t) = \{x \in \mathbb{R}^2_+ | x^1 + 3x^2 = t$, where $x^1 \geq \frac{t}{4}$ or $3x^1 + x^2 = t$, where $x^1 \leq \frac{t}{4}\}$ when $t \in [4, +\infty)$. Let $W : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a set-valued correspondence such that $W(x) = \{t | x \in q(t)\}$. Notice that $W(x)$ is always single-valued except for $W(1, 1) = [2, 4]$. For each $E = (N, u_N, \omega)$, let $\bar{\mu}(E) = \max\{x \in A^*(E) | \prod_{i\in N} I_{W(x_i)}(t) \neq 0$ for some $t \in \mathbb{R}\}$ where $I_{W(x_i)}(t) = 1$ when $t \in W(x_i)$, and $I_{W(x_i)}(t) = 0$ when $t \in \mathbb{R} \setminus W(x_i)$ for all $i$. It is a well-defined rule, satisfies all the axioms except for resource monotonicity. For a counter-example, consider a preference profile $(u_1, u_2)$ such that $\gamma_2$ contains $x_2 = (1, 1)$, as shown in Figure 10. Let $x_1 \in \gamma_1 \cap q(2)$ and $x'_1 \in \gamma_1 \cap q(t)$ for some $t \in (2, 4)$. Then $\bar{\mu}(x_1 + x_2) = (x_1, x_2)$, $\bar{\mu}(x'_1 + x_2) = (x'_1, x_2)$. In this case, $x_1 + x_2 < x'_1 + x_2$ but agent 2 is not better off. Note that this rule still satisfies the second version of resource monotonicity.

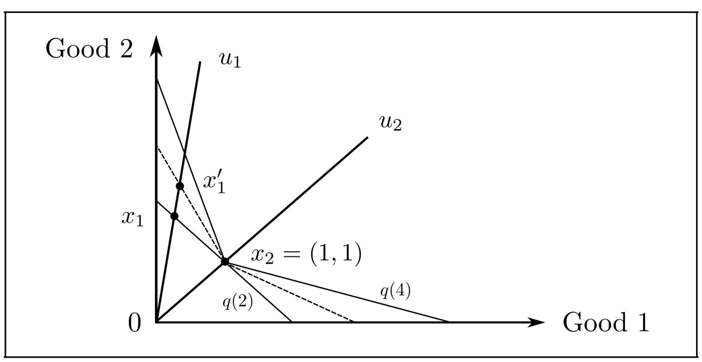


Figure 1.10 : Tightness of resource monotonicity

Drop the consistency, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \mu^{W_1}(E)$ if $|N|$ is even and $\bar{\mu}(E) = \mu^{W_2}(E)$ if $|N|$ is odd where $W_1$ and $W_2$ are as in Example 1 with different $p$'s. Obviously, $\bar{\mu}$ is EFFN, RM, ANON, GSP and EF, but not CST.

Drop the anonymity, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$ with $1 \notin N$, $\bar{\mu}(E) = \mu^W(E)$ where $W$ is as in Example 1, and for all $E = (N, u_N, \omega)$ with $1 \in N$, $\bar{\mu}(E) = \max\{x \in A^*(E) | 2W(x_1) = W(x_i),\ \forall i \in N \setminus \{1\}\}$. It is a well-defined rule, and is EFFN, RM, CST, GSP, but not ANON. We will prove this result for a general class of such rules in the next section.

Drop the strategy-proofness (and thus the group strategy-proofness), and consider the following rule $\bar{\mu}$ which is EFFN, ANON, RM and CST. Let $\bar{u} \in \tilde{\mathcal{U}}$ be fixed. For all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \mu^W(E)$ if $\forall i \in N$, $u_i \neq \bar{u}$, and if $S = \{j \in N | u_j = \bar{u}\} \neq \emptyset$, $\bar{\mu}(E) = \max\{x \in A^*(E) | 2W(x_j) = W(x_i),\ j \in S,\ i \in N \setminus S\}$ where $W$ is as in Example 1. It is easy to check that $\bar{\mu}$ is well-defined and satisfies the above axioms. Figure 11 illustrates that $\bar{\mu}$ is not SP (and thus not GSP) in a two-commodity space. Consider a two-agent economy where their utility profile is as given in Figure 11. Suppose that $\bar{\mu}(\{1, 2\}, (u_1, u_2), \omega) = (x_1, x_2)$ for some $\omega \in \mathbb{R}^2_+$. Then agent 1 prefers to report $u'_1$ which is very "close" to $u_1$. The point on $\gamma'_1$ "moves" faster than on $\gamma_1$, so after agent 1's misreport, it must be that his allocated bundle $x'_1 > x_1$.

Drop the envy-freeness, the above two rules also work as counter-examples. This is because envy-freeness implies anonymity and strategy-proofness when a rule is EFFN, RM and CST.

## 1.7 Agent-specific Egalitarian Rules

Now we consider a natural extension of $\tilde{\mathcal{M}}$ to a class of non-anonymous rules. While generalized egalitarian rules equalize the agents' final welfare levels according to a benchmark

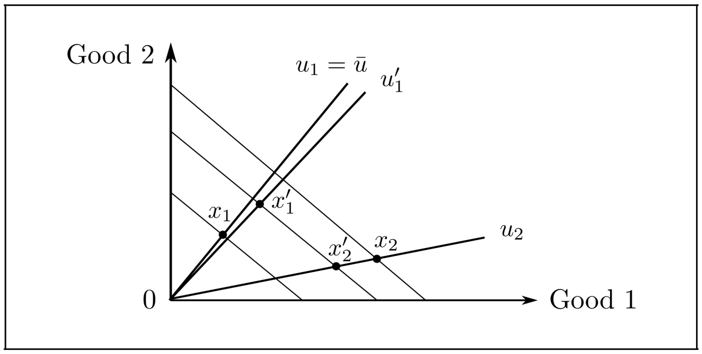


Figure 1.11 : Tightness of strategy-proofness

preference over the commodity space, society may measure the welfare of each agent differently. It may attach to each agent $i$ a utility function $W_i$ and equalize the agents' final welfare according to these agent-specific utility functions.

Formally, for all $i \in \mathbb{N}$, let $W_i : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ be a strictly increasing continuous function such that $W_i(\mathbf{0}) = 0$. Let $\mathcal{W}^a = \{W_i | i \in \mathbb{N}\}$ be a set of all agents' welfare indices.

**Definition 7.** *A rule $\mu$ is called an agent-specific egalitarian rule if there exists $\mathcal{W}^a$ such that for all $E \in \mathcal{E}$,*

$$\mu(E) = \max\{x \in A^*(E) | W_i(x_i) = W_j(x_j),\ \forall i, j \in N\}$$

*where $W_i \in \mathcal{W}^a$, $\forall i \in N$. Let $\mathcal{M}^a$ denote the class of agent-specific egalitarian rules.*

Using the similar argument as in the proof of Lemma 2, it is easy to see that the analogous result holds, and $\mathcal{M}^a$ is well-defined.

**Theorem 3.** *If $\mu$ is in $\mathcal{M}^a$, then $\mu$ is efficient, resource monotonic, consistent and group strategy-proof.*

*Proof.* The proof is almost the same as what we did for generalized egalitarian rules. Just by replacing $W(x_i)$ with $W_i(x_i)$ in Step 4 and 5 of Section 5, we can get the desired results.

□

## 1.8 Endowment-specific Egalitarian Rules and Private Property

Another extension of $\tilde{\mathcal{M}}$ is natural when we drop the common property assumption. We first introduce the model where every agent has a private endowment. For notational simplicity, we will abuse the previous symbols again to denote the corresponding notions in the model with private property.

An economy $E$ is a triple $(N, u_N, \omega_N)$ where $N \subseteq \mathbb{N}$ is a nonempty finite set of agents, $u_N = (u_i)_{i \in N}$ with $u_i \in \tilde{\mathcal{U}}$, $\forall i \in N$, is a preference profile, and $\omega_N = (\omega_i)_{i \in N}$ with $\omega_i \in \mathbb{R}^l_+$, $\forall i \in N$, denotes a vector of private endowments of the agents. Let $\mathcal{E}$ be the set of all economies.

Given $(N, \omega_N)$, the set of all feasible allocations is $A(N, \omega_N) = \{x \in \mathbb{R}_+^{|N| \times l} | \sum\limits_{i \in N} x_i \leq \sum\limits_{i \in N} \omega_i\}$. For any economy $E = (N, u_N, \omega_N)$, the set of non-wasteful feasible allocations is $A^*(E) = A(N, \omega_N) \cap \prod\limits_{i \in N} \gamma_i$ where $\gamma_i$ is the critical set of $u_i$. Let $\mathcal{A}^* = \{A^*(E) | E \in \mathcal{E}\}$. A rule is a mapping $\mu : \mathcal{E} \to \mathcal{A}^*$ such that $\mu(E) \in A^*(E)$ for all $E \in \mathcal{E}$.

When the private property is introduced, an important problem is whether the agents are willing to put their own endowments together and participate in the social reallocation. Hence, here we need the individual rationality axiom to guarantee the voluntary participation.

A rule $\mu$ is *individually rational* (IR) if $\forall (N, u_N, \omega_N)$, $\forall i \in N$, $u_i(\mu_i(N, u_N, \sum\limits_{i \in N} \omega_i)) \geq u_i(\omega_i)$.

The efficiency, incentive compatibility and fairness axioms are defined in the same way as the previous ones, except a little modification on anonymity and resource monotonicity.

Let $\pi$ be a bijection on $\mathbb{N}$. A rule $\mu$ is *anonymous* if $\forall \pi$, $\forall (N, u_N, \omega_N)$, $\forall i \in N$, $\mu_i(N, u_N, \omega_N) = \mu_{\pi(i)}(\pi(N), (u_{\pi(j)})_{\pi(N)}, (\omega_{\pi(j)})_{\pi(N)})$ where $u_j = u_{\pi(j)}$ and $\omega_j = \omega_{\pi(j)}$, $\forall j \in N$.

A rule $\mu$ is *resource monotonic* if $\forall (N, u_N)$, $\forall \omega_N, \omega'_N \in \mathbb{R}_+^{|N| \times l}$, $\omega_i > \omega'_i$ for all $i \in N$ implies that $u_i(\mu_i(\omega_N)) > u_i(\mu_i(\omega'_N))$.

Resource monotonicity is shown to be incompatible with efficiency and individual rationality in Moulin and Thomson (1988). Although they assume a larger preference domain and use another version of resource monotonicity, it is easy to check that with a slight modification their counter-example still works in our context.

Our last result shows that when we allow the welfare index of an agent to depend on his private endowment, we obtain a class of rules which is EFFN, GSP, ANON and IR.

For all $x \in \mathbb{R}^l_+$, let $W_x : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ be a strictly increasing and continuous function such that for all $y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$ with $y \leq x$ and $y^k = x^k$ for some $k \in L$, $W_x(y) = 1$.[8] Let $\mathcal{W}^e = \{W_x | x \in \mathbb{R}^l_+\}$.

**Definition 8.** *A rule $\mu$ is called an endowment-specific egalitarian rule, if there exists $\mathcal{W}^e$ such that for all $E \in \mathcal{E}$,*

$$\mu(E) = \max\{x \in A^*(E) | W_{\omega_i}(x_i) = W_{\omega_j}(x_j),\ \forall i, j \in N\}$$

*where $W_{\omega_i} \in \mathcal{W}^e$, $\forall i \in N$. Let $\mathcal{M}^e$ denote the class of endowment-specific egalitarian rules.*

By the analogous result of Lemma 2, $\mathcal{M}^e$ is well-defined.

**Theorem 4.** *If a rule $\mu$ is in $\mathcal{M}^e$, then it is efficient, group strategy-proof, anonymous and individually rational.*

*Proof.* The proof of efficiency, group strategy-proofness and anonymity is basically the same as in Step 4 and 5 of Section 5.

To see that $\mu$ is IR, note that for all $E = (N, u_N, \omega_N)$, there exists the allocation $x \in A^*(E)$ such that $\forall i \in N$, $u_i(x_i) = u_i(\omega_i)$ and $W_{\omega_i}(x_i) = 1$. Hence, $\forall i \in N$, $\mu_i(E) \geq x_i$ and then $u_i(\mu_i(E)) \geq u_i(\omega_i)$. □

[8] Essentially, what we need is that $W_x(y)$ is some constant which is independent of $x$.

## 1.9 Concluding Remarks

In this paper, we study fair allocation rules on the generalized Leontief preference domain and achieve very positive results. Nevertheless, there are still some immediate open questions. The characterization of the agent-specific and endowment-specific egalitarian rules remains open. Another intriguing question is how we could drop the non-wastefulness assumption of the rules and still get some positive results. We also observe that recently de Castro et al. (2011) find nice properties of consumption allocation in asymmetric information economies under Maximin preferences, which has some structural resemblance to Leontief preferences without uncertainty. We would like to investigate the relationship between the two problems in the future.

## 1.10 Appendix

**1. The proofs of the results in Section 3**

**Lemma 5.** *If* $\succsim \in \mathcal{D}$*, then*

*(i)* $\succsim$ *is monotone, i.e.,* $\forall x, y \in \mathbb{R}^l_+$, $x > y$ *implies that* $x \succ y$*;*

*(ii) for any* $x \in \mathbb{R}^l_+$, $U_\succsim(x) = \{a\} + \mathbb{R}^l_+$ *implies that* $I_\succsim(x) = \{a\} + \partial\mathbb{R}^l_+$.

*Proof.* Let $\succsim \in \mathcal{D}$ be given.

(i) Suppose that $x, y \in \mathbb{R}^l_+$ and $x > y$. Since $\succsim$ is locally non-satiated, we can find $y' < x$ such that $y' \succ y$. Let $U_\succsim(y') = \{a\} + \mathbb{R}^l_+$, $a \in \mathbb{R}^l_+$. Since $y' \in U_\succsim(y')$ and $x > y'$, then $x \geq a$, and thus $x \in U_\succsim(y')$. Hence, $x \succsim y' \succ y$.

(ii) Suppose that $x \in \mathbb{R}^l_+$ and $U_\succsim(x) = \{a\} + \mathbb{R}^l_+$. By (i), $\forall y \in \{a\} + \mathring{\mathbb{R}}^l_+$, $y \succ x$. Now let $y \in \{a\} + \partial\mathbb{R}^l_+$. Since $\succsim$ is continuous, if $y \succ x$, then there exists $y' < y$ such that $y' \succ x$, which contradicts that $U_\succsim(x) = \{a\} + \mathbb{R}^l_+$. Hence, $I_\succsim(x) = \{a\} + \partial\mathbb{R}^l_+$. □

**Lemma 6.** *For any* $\succsim \in \mathcal{D}$*,* *(i)* $\mathbf{0} \in \gamma_{\succsim}$*, and* $\gamma_{\succsim}$ *is unbounded;*

*(ii) if* $a, b \in \gamma_{\succsim}$ *and* $a \neq b$*, then either* $a < b$ *or* $a > b$*, i.e.,* $\gamma_{\succsim}$ *is totally ordered;*

*(iii)* $\gamma_{\succsim}$ *is connected;*

*(iv)* $\gamma_{\succsim}$ *is closed.*

*Proof.* Let $\succsim \in \mathcal{D}$ be given.

(i) To see $\mathbf{0} \in \gamma_{\succsim}$, it suffices to show that $U_{\succsim}(\mathbf{0}) = \{\mathbf{0}\} + \mathbb{R}^l_+$. Suppose the contrary that $U_{\succsim}(\mathbf{0}) = \{a\} + \mathbb{R}^l_+$ where $a \neq \mathbf{0}$. Then it implies that $\mathbf{0} \notin U_{\succsim}(\mathbf{0})$, a contradiction.

For unboundedness, suppose the contrary that there exists $y \in \mathbb{R}^l_+$ such that $\forall a \in \gamma_{\succsim}$, $a < y$. Suppose $U_{\succsim}(y) = \{b\} + \mathbb{R}^l_+$. Then $b \in \gamma_{\succsim}$ and $I_{\succsim}(y) = \{b\} + \partial\mathbb{R}^l_+$. Thus $b \leq y$ and $y^k = b^k$ for some $k \in \{1, ..., l\}$, which is a contradiction.

(ii) Let $a, b \in \gamma_{\succsim}$ and $a \neq b$. Suppose that $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ and $U_{\succsim}(y) = \{b\} + \mathbb{R}^l_+$, $x, y \in \mathbb{R}^l_+$. It is not true that $x \sim y$, otherwise $a = b$. By Lemma 5 (ii), $a \sim x$ and $b \sim y$. If $x \succ y$, then $a \succ y$ and thus $a \in \{b\} + \mathring{\mathbb{R}}^l_+$, which means $a > b$. Similarly, if $y \succ x$, then $a < b$.

(iii) Define $\rho : \gamma_{\succsim} \to \mathbb{R}_+$ such that $\rho(x) = \sum_{k \in L} x^k$, $\forall x \in \gamma_{\succsim}$. It suffices to show that $\rho$ is a homeomorphism.

The injectivity of $\rho$ follows from (ii). We first prove that $\rho$ is surjective. Suppose the contrary that there exists $t \in \mathbb{R}_+ \setminus \rho(\gamma_{\succsim})$. Then $\gamma_{\succsim} = \alpha \cup \beta$ where $\alpha = \{a \in \gamma_{\succsim} | \rho(a) < t\}$ and $\beta = \{b \in \gamma_{\succsim} | \rho(b) > t\}$. By (i) we know that $\rho(\mathbf{0}) = 0$, and $\sup \rho(\gamma_{\succsim}) = \infty$. Hence, $\alpha, \beta \neq \emptyset$. Let $\bar{a} = \sup \alpha$ and $\underline{b} = \inf \beta$. Clearly, $\bar{a}, \underline{b} \in \mathbb{R}^l_+$ and $\bar{a} \leq \underline{b}$. If there exists $h \in L$ such that $\bar{a}^h < \underline{b}^h$, then pick $x \in \mathbb{R}^l_+$ such that $\bar{a} < x$ and $x^h < \underline{b}^h$. Suppose $I_{\succsim}(x) = \{c\} + \partial\mathbb{R}^l_+$. Thus $c \in \beta$ and $x \geq c$, which contradicts that $x^h < \underline{b}^h$. Hence, $\bar{a} = \underline{b}$. Then by (ii), $I_{\succsim}(\bar{a}) = \{\bar{a}\} + \partial\mathbb{R}^l_+$. Thus, either $\bar{a} \in \alpha$ or $\bar{a} \in \beta$. If $\bar{a} \in \alpha$, then $\rho(\bar{a}) < t$. We can choose $b \in \beta$ such that $\rho(b)$ is arbitrarily close to $\rho(\bar{a})$, and this contradicts that $\rho(b) > t$. Similarly, if $\bar{a} \in \beta$, we can also get a contradiction.

Next observe that for any $x, y \in \gamma_{\succsim}$, $\|x - y\| \leq |\rho(x) - \rho(y)| \leq l\|x - y\|$ [9], since either $x < y$ or $x > y$. Hence, $\rho$ is a continuous open mapping.

(iv) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in $\gamma_{\succsim}$ such that $\lim_{n\to\infty} a_n = a$. If $a \notin \gamma_{\succsim}$, then $\gamma_{\succsim} = [\gamma_{\succsim} \cap (\{a\} + \mathbb{R}^l_+)] \cup [\gamma_{\succsim} \cap (\{a\} - \mathbb{R}^l_+)]$, since $\gamma_{\succsim}$ is totally ordered and $a$ is the limit of a sequence of elements in $\gamma_{\succsim}$. This contradicts that $\gamma_{\succsim}$ is connected. □

**Proposition 1.** *For any $\succsim \in \mathcal{D}$, $\succsim$ is represented by $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, where $\zeta : \mathbb{R}_+ \to \gamma_{\succsim}$ is a strictly increasing homeomorphism such that $\sum_{k\in L} \zeta^k(t) = t$, $\forall t \in \mathbb{R}_+$.*

*Proof.* Let $\succsim \in \mathcal{D}$ be given. Suppose that $\rho$ is defined as in the proof of Lemma 6 (iii). Clearly, $\rho$ is strictly increasing since $\gamma_{\succsim}$ is totally ordered. Let $\zeta = \rho^{-1}$. Hence, all the properties of $\zeta$ follows from those of $\rho$. Since $\zeta(\mathbb{R}_+)$ is unbounded and $\zeta$ is continuous, then $\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$ is bounded and closed for any $x \in \mathbb{R}^l_+$, and thus $u : \mathbb{R}^l_+ \to \mathbb{R}_+$ is well defined.

Now we show that $u$ represents $\succsim$. If $x \sim y$ and $I_{\succsim}(x) = \{a\} + \partial\mathbb{R}^l_+$, then $u(x) = u(y) = \sum_{k\in L} a^k$, since $\zeta$ is strictly increasing. If $x \succ y$, $I_{\succsim}(x) = \{a\} + \partial\mathbb{R}^l_+$ and $I_{\succsim}(y) = \{b\} + \partial\mathbb{R}^l_+$, then by Lemma 6 (ii) and Lemma 5 (i), $a > b$. Thus $u(x) = \sum_{k\in L} a^k > \sum_{k\in L} b^k = u(y)$. □

## 2. The relaxation of the unbounded allocation assumption

There are several places in the steps of the proofs to be modified when we drop the assumption that $\forall (N, u_N)$, $\forall i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+)$ is unbounded.

**Step 1.** (ii) Suppose that $\mu$ is EFFN and RM. If for $(N, u_N)$ and $i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+)$ is bounded, then $\mu_i(N, u_N, \mathbb{R}^l_+) = \{x_i \in \gamma_i | x_i < x_i^*\}$ for some $x_i^* \in \gamma_i$, and moreover, there exists $j \in N$ such that $\mu_j(N, u_N, \mathbb{R}^l_+) = \gamma_j$.

[9] $\|\cdot\|$ is the standard Euclidean norm.

*Proof.* Let $(N, u_N)$ and $i \in N$ be given. Suppose that $\mu_i(\mathbb{R}^l_+)$ is bounded. Let $x^*_i = \sup \mu(\mathbb{R}^l_+)$. Since $\gamma_i$ is closed, then $x^*_i \in \gamma_i$. Note that if $x_i \in \mu_i(\mathbb{R}^l_+)$, then $x_i + \epsilon \in \mu_i(\mathbb{R}^l_+)$ for some $\epsilon > \mathbf{0}$, since $\mu$ is RM. Hence, $x^*_i \notin \mu_i(\mathbb{R}^l_+)$. Then using the similar argument as in the proof of Step 1, we get that $\forall x_i \in \gamma_i$ such that $x_i < x^*_i$, $x_i \in \mu_i(\mathbb{R}^l_+)$. If $\forall i \in N$, $\mu_i(\mathbb{R}^l_+)$ is bounded, then pick $\omega \geq \sum_{i \in N} x^*_i$, and thus $\sum_{i \in N} \mu_i(\omega) < \sum_{i \in N} x^*_i \leq \omega$, which contradicts that $u$ is EFFN. □

**Step 3.** The sufficiency part.

*Proof.* Let all the assumptions as in the sufficiency proof of Step 3 be given. we only need to check the case when there does not exist $y_1 \in \gamma'_1$ such that $(y_1, x_2) \in \mu(\{1, 2\}, (u'_1, u_2), \mathbb{R}^l_+)$. Pick $y'_1 \in \gamma'_1$ such that $y'_1 > x_1$. By the modified Step 1 (ii), we can find $y_2 \in \gamma_2$ such that $(y'_1, y_2) \in \mu(\{1, 2\}, (u'_1, u_2), \mathbb{R}^l_+)$, and $y_2 < x_2$. Since $\mu$ is CST, $(x_1, x_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. Again by the modified Step 1, there exist $y_1 \in \gamma_1$ such that $(y_1, y_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. Since $y_2 < x_2$, then $y_1 < x_1$, and thus $y_1 < y'_1$, which contradicts our assumption. □

**Step 4.** The sufficiency part.

*Proof.* We first show the following two statements:

(i) If $\mu$ is EFFN, RM and ANON, then $\forall (N, u_N)$ such that $\forall i \in N$, $u_i = u$, $\mu_i(N, u_N, \mathbb{R}^l_+) = \gamma_i$, $\forall i \in N$;

(ii) If $\mu$ is EFFN, RM, ANON and SP, then $\forall (N, u_N)$ such that $|N| = 2$ and $\gamma_i$ is unbounded in every commodity for some $i \in N$, $\mu_j(N, u_N, \mathbb{R}^l_+) = \gamma_j$ where $j \in N$ and $j \neq i$.

The result (i) follows from Remark 1 and the modified Step 1.

For (ii), let $(N, u_N)$ which satisfies the required conditions be given. By the modified Step 1, suppose the contrary that $\mu_j(\mathbb{R}^l_+)$ is bounded where $j \in N$ and $j \neq i$. Thus when $\omega$ is big enough, agent $j$ would pretend to have agent $i$'s preference, since his allocation would be unbounded in every dimension by statement (i) and the assumption on $\gamma_i$. This contradicts that $\mu$ is SP.

Then the construction of $W$ is basically the same except that $\bar{u}$ should be chosen such that its critical set is unbounded in every dimension. By statement (ii), $W$ is well-defined. The rest of the proof is the same.

$\square$

# Chapter 2

# Three representations of preferences with decreasing absolute uncertainty aversion

## 2.1 Introduction

It is a well-known economic phenomenon that people's risk aversion decreases with wealth. For example, consider a risk of winning or losing \$500 with equal probability. A person with initial wealth \$550 should be willing to pay more for insurance than should a person with wealth \$50,000. In this case, his preference is said to display *decreasing absolute risk aversion* (Arrow (1963), Pratt (1964)).

When applying this theory to macroeconomics and finance models, researchers typically assume either that the risk is objectively given, or that the decision maker has a subjective probability. However, empirical evidence shows that in real-life settings uncertainty is more relevant than risk.[1] That is, the probability distribution governing the potential outcomes is often unknown to the decision maker (Knight, 1921). Moreover, although the subjective probability assumption finds its axiomatic foundation in Savage (1954), Ellsberg (1961)'s famous thought experiments and many subsequent field experiments reveal that people's behavior violates Savage's axioms, suggesting that decision makers do not have a subjective probability.

Nevertheless, the wealth effect on uncertainty has not been sufficiently studied. Most

[1]For example, an investor chooses the optimal amount of investment in the presence of unknown random shocks that impinge on production, people decide how much to spend on health insurance without knowing exactly the chance of getting sick, and the environmental department makes regulations for a new technology based only on an estimation of the pollution rate.

commonly used models of preferences under uncertainty impose the restrictive assumption that the degree of a decision maker's uncertainty aversion is constant. However, it might be expected that wealthier people are more willing to take uncertainty-bearing behavior. If in the opening example the probability of winning and losing is unknown to the decision maker, he may still be willing to pay more for insurance when he has initial wealth \$550 than when he has initial wealth \$50,000. In this case, the decision maker's preference exhibits *decreasing absolute uncertainty aversion*.

This paper studies the effect of wealth on uncertainty aversion. Our first main result axiomatizes a class of preferences that display decreasing absolute uncertainty aversion. Three equivalent representations are obtained.[2]

All axioms considered in this paper are standard in the literature, with one important innovation — an axiom called *decreasing absolute uncertainty aversion* (hereafter DAUA).[3] Consider an act $f$ as a state-contingent payoff profile, and call an act constant if it gives the same payoff in each state. The DAUA axiom requires that if an act $f$ is weakly preferred to a constant act, then it is still weakly preferred after a common improvement in every state for both acts. This implies that a decision maker becomes weakly more tolerant to the uncertainty of $f$ as he gets wealthier. The DAUA axiom weakens Gilboa and Schmeidler (1989)'s certainty independence axiom and Maccheroni, Marinacci and Rustichini (2006)'s weak certainty independence axiom. Preferences satisfying our axioms are called *DAUA variational preferences*. DAUA variational preferences include several important classes of preferences which display constant absolute uncertainty aversion (see the second main result).

We obtain three different yet equivalent representations for DAUA variational

[2]The results for preferences with increasing absolute uncertainty aversion are analogously obtained.

[3]Note that throughout the paper, "decreasing absolute uncertainty aversion" means "non-increasing absolute uncertainty aversion".

preferences.[4] The first representation is a *variant constraint representation*:

$$V(f) = \min_{p \in \{p \in \Delta | d(p,B) \leq \eta(u(f))\}} E_p u(f). \tag{2.1}$$

It models situations in which the decision maker has a set $B$ of best-guess priors but does not fully trust these priors. He considers all priors $p$ within $\eta$ distance of $B$, and evaluates $f$ by its minimum expected utility over such neighborhood of $B$. The bound constraint $\eta$ is a function of utility profiles. In particular, it weakly decreases in the ensured (or baseline) utility level of an act. This means that the decision maker becomes less concerned with robustness — equivalently, more tolerant to uncertainty — as he becomes better off overall.

The variant constraint representation in (2.1) is a variation on the *constraint criterion* introduced by Hansen and Sargent (2001) as a robust decision rule. A constraint criterion evaluates an act by

$$V(f) = \min_{p \in \{p \in \Delta | R(p \| q) \leq \eta\}} E_p u(f),$$

where $q$ is a best-guess prior and $R(p\|q)$ is the relative entropy of $p$ with respect to $q$. An important difference when compared to (2.1) is that here, $\eta$ is constant over all the utility profiles, which implies that the degree of the decision maker's uncertainty aversion is *fixed*. While an axiomatization of the constraint criterion is still an open question, our result provides an axiomatic foundation for a variant constraint criterion which is of the same spirit and allows decreasing absolute uncertainty aversion.

[4]A representaion for a preference $\succsim$ over acts is a function $V$ of acts such that $f \succsim g \Leftrightarrow V(f) \geq V(g)$ for all acts $f, g$.

The second representation is a *weighted maxmin representation*:

$$V(f) = \lambda(u(f)) \min_{p \in C} E_p u(f) + (1 - \lambda(u(f))) \max_{p \in C} E_p u(f). \tag{2.2}$$

The decision maker considers $C$ as the set of all possible priors and evaluates an act by a weighted average of the minimum and maximum expected utility over $C$. The weight $\lambda$ on the worst case weakly decreases in the baseline utility, meaning that the decision maker becomes more optimistic (less uncertainty averse) with the increase of his ensured payoffs.

The third representation is the *DAUA variational representation*:

$$V(f) = \min_{p \in \Delta} [E_p u(f) + c(E_p u(f), p)], \tag{2.3}$$

where $c$ is a cost function of expected utilities and priors. The cost function basically plays the role of restricting the priors under consideration. In particular, $c$ weakly increases in the utility term. It turns out that when the baseline utility of an act rises, it will be evaluated by a more "favorable" prior. When $c$ is constant in utility, (2.3) is reduced to Maccheroni, Marinacci and Rustichini (2006)'s variational representation, which represents the class of variational preferences that display constant absolute uncertainty aversion.

The three representations relate to each other in a nice way. The set of the priors with zero cost at any utility level is exactly the set $B$, while the set $C$ contains precisely the priors that have finite cost at some utility level. The set $B$ is a subset of $C$. They provide, respectively, an upper and lower bound for the evaluation. The value of each act is below the worst expected utility over $B$ and above that over $C$. Moreover, the bounds are tight. At best, the decision maker considers only the priors in $B$; at worst, he considers all the priors in $C$. Gilboa and Schmeidler (1989)'s maxmin preferences are exactly characterized by the condition $B = C$.

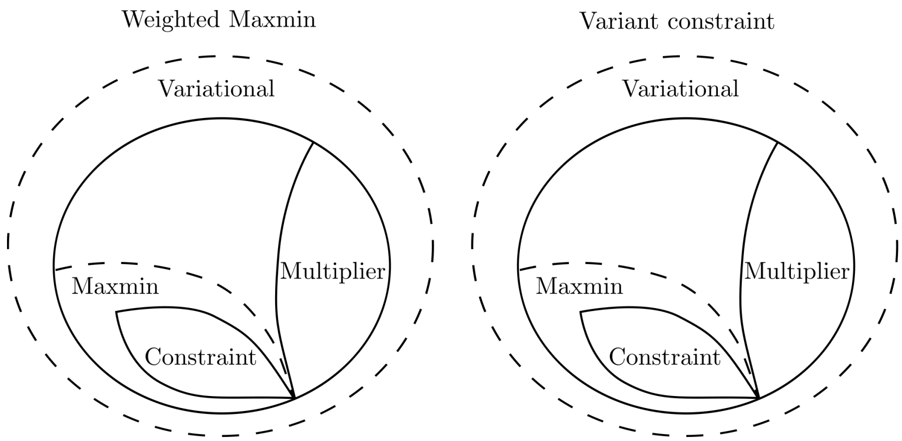


Figure 2.1 : Relations of preferences that display constant absolute uncertainty aversion

Our second main result is to establish relationships among the representations for several important classes of preferences in the literature, when we restrict our attention to preferences displaying constant absolute uncertainty aversion. One finding is that the representations for three nested classes of preferences — variational, maxmin and constraint preferences — in fact “commute" with each other. More precisely, while Maccheroni, Marinacci and Rustichini (2006) build the connection between maxmin preferences (and thus its subclass, constraint preferences) and multiplier preferences by showing that both belong to a larger class of variational preferences, our result suggests that the converse is true as well. That is, variational preferences also live in a class of generalized maxmin preferences and in a class of a constraint type of preferences (see Figure 1).

This result comes from the following observation. The subclass of DAUA variational preferences with constant absolute uncertainty aversion is exactly the class of variational preferences. The three representations above, when restricted to variational preferences, also give three equivalent representations. In the first two representations, the distance constraint $\eta$ and the weight function $\lambda$ become constant in the baseline utility of an act. Representation (2.3), with the cost function $c$ no longer depending on the utility term, is reduced to the variational representation obtained by Maccheroni, Marinacci and Rustichini (2006).

The other finding is an equivalent representation for the multiplier criterion[5] introduced by Hansen and Sargent (2001) as the second robust decision rule. This representation closely resembles the constraint criterion and clearly shows the relationships between the two criteria. Hansen and Sargent (2001) establish the "equivalence" between constraint and multiplier criteria in a dynamic resource allocation problem by showing that both rules imply the same optimal solution. However, they generally give different rankings of acts other than the optimal one. we further clarify their relationship by an equivalent representation for multiplier criterion:

$$V(f) = \min_{p \in \{p \in \Delta | R(p\|q) \leq \eta(u(f))\}} E_p u(f),$$

where $\eta$ is a function of utility profiles corresponding to the parameter in the multiplier criterion. This shows that the difference between the constraint and multiplier criteria lies in the distance constraint $\eta$. For the constraint criterion, $\eta$ is a constant function. For the multiplier criterion, $\eta$ is a particular function which is constant only if it is constantly 0.

This paper is organized as follows. Section 2 states the axioms. Section 3 presents the three representations. Section 4 studies several subclasses of DAUA variational preferences and provides another representation for multiplier preferences. Section 5 concludes.

### 2.1.1 Related literature

Although decreasing absolute uncertainty aversion is a natural analogy of the classic concept of decreasing absolute risk aversion, there are only two recent papers addressing this issue. Klibanoff, Marinacci and Mukerji (2005) characterize a class of preferences such

[5]The multiplier criterion evaultes an act by $V(f) = \min_{p \in \Delta}[E_p u(f) + \theta R(p\|q)]$, where $\theta \in (0, \infty]$ is a parameter.

that a decision maker has a von Neumann-Morgenstern utility over the outcomes and treats each act as a prior-contingent expected utility function. The decision maker has a second-order belief over all priors and evaluates an act by the expectation of an increasing transformation of its prior-dependent expected utility function. The expectation is taken with respect to his second-order belief over priors, and the increasing transformation is viewed as a second-order utility function. They characterize preferences with decreasing absolute uncertainty aversion by the properties of a second-order utility function, analogous to the approach used with decreasing absolute risk aversion.

Chambers, Grant, Polak and Quiggin (2012) study a two-parameter model where a decision maker has a baseline prior and a measure of dispersion of acts. The decision maker evaluates an act based on its mean with respect to the baseline prior and its dispersion. They represent preferences with decreasing absolute uncertainty aversion by the property of an aggregating function of mean and dispersion. Their DAUA axiom is stronger than the one in this paper. Their axiom compares the effects of improving a certainty part on all pairs of acts where one act is more dispersed than the other.

Our paper adopts a different approach and studies a different model of preferences with decreasing absolute uncertainty aversion. This approach accommodates situations where the decision maker does not have a second-order belief or a baseline prior, but only a range of estimated priors.

## 2.2 Setup

We denote by $\mathbb{R}$ the set of all the reals, and $\mathbb{R}_+$ the set of all the non-negative reals. Let $S$ be a set of *states of the world*. A subset of $S$ is called an *event*. We assume that $S$ is finite and has cardinality $n$. The set of all probabilities on $S$ is denoted by $\Delta$. We identify $\Delta$ with the unit simplex in $\mathbb{R}^n$, i.e., the set $\{(p_1, ..., p_n) \in \mathbb{R}^n | \sum_{i=1}^{n} p_i = 1 \text{ and } p_i \geq 0 \text{ for all } i\}$, and $\Delta$ is regarded as a metric space with the Euclidean metric.

Let $X$ be a set of *outcomes*. We follow Maccheroni, Marinacci and Rustichini (2006) to assume that $X$ is a convex subset of some vector space. Note that it includes Anscombe and Aumann (1963)'s classic setting where $X$ is the set of all lotteries on a set of prizes. An *act* is a function $f : S \to X$. Let $\mathcal{F} = X^S$ be the set of all acts. Given an outcome $x \in X$, with a slight abuse of notation, we also denote by $x$ the *constant act* which assigns $x$ to all $s \in S$, and identify $X$ with the set of all constant acts. Given $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we define the convex combination $\alpha f + (1 - \alpha)g$ as an act in $\mathcal{F}$ such that $[\alpha f + (1 - \alpha)g](s) = \alpha f(s) + (1 - \alpha)g(s)$ for all $s \in S$.

A decision maker's *preference* is a binary relation $\succsim$ on $\mathcal{F}$. Let $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$ as usual. Given $f \in \mathcal{F}$, an element $x_f \in X$ is a *certainty equivalent* of $f$ if $x_f \sim f$. A function $V : \mathcal{F} \to \mathbb{R}$ *represents* $\succsim$ on $\mathcal{F}$ if $f \succsim g \Leftrightarrow V(f) \geq V(g)$ for all $f, g \in \mathcal{F}$.

## 2.3 Axioms

Consider the following axioms for $\succsim$.

**A.1. Weak Order.** (1) For all $f, g \in \mathcal{F}$, either $f \succsim g$ or $g \succsim f$.

(2) For all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

**A.2. Decreasing Absolute Uncertainty Aversion.** For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in (0, 1)$, if either $f$ is a constant act or $y \succsim x$, then

$$\begin{aligned} &\alpha f + (1 - \alpha)x \succsim \alpha z + (1 - \alpha)x \\ &\qquad \Rightarrow \alpha f + (1 - \alpha)y \succsim \alpha z + (1 - \alpha)y. \end{aligned}$$

**A.3. Continuity.** For all $f, g, h \in \mathcal{F}$, the set $\{\alpha \in [0,1] | \alpha f + (1-\alpha)g \succsim h\}$ and the set $\{\alpha \in [0,1] | h \succsim \alpha f + (1-\alpha)g\}$ are closed in $\mathbb{R}$.

**A.4. Monotonicity.** For all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for every $s \in S$, then $f \succsim g$.

**A.5. Uncertainty Aversion.** For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \sim g$, then $\alpha f + (1-\alpha)g \succsim f$.

**A.6. Unboundedness.** There exist $x, y \in X$ such that (1) $x \succ y$, and (2) for each $\alpha \in (0,1)$, there are $z, z' \in X$ satisfying $\alpha z + (1-\alpha)y \succsim x \succ y \succsim \alpha z' + (1-\alpha)x$.

A preference $\succsim$ on $\mathcal{F}$ is called a *DAUA variational preference* if it satisfies Axiom A.1 - A.6.

Axiom A.1, A.3, A.4 and A.5 are standard in the literature (see e.g. Anscombe and Aumann (1963), Schmeidler (1989) and Gilboa and Schmeidler (1989)). Weak order requires preferences to be complete and transitive. Continuity states that preferences are continuous with respect to the coefficients of convex combination of acts. Monotonicity assumes that the decision maker ranks the outcomes as constant acts, and that an act is weakly preferred if it assigns a weakly better outcome in each state. Uncertainty aversion captures the decision maker's preference for hedging under uncertainty.

Axiom A.6 is stronger than the usual non-degeneracy axiom. The non-degeneracy axiom asks that there exists at least one act which is strictly preferred to some other. Axiom A.6 enforces the obtained utility function on $X$ representing $\succsim$ on constant acts to range over all the reals. This axiom is commonly used in the recent literature (see e.g. Kopylov (2001), Maccheroni, Marinacci and Rustichini (2006), Strzalecki (2011b) and Grant and Polak (2011)). In some places it is a technical assumption which simplifies the analysis, while in the other places it is indispensable for some desirable results. In this paper, A.6 is necessary since our representation crucially relies on the preference for the "limiting acts", i.e., the acts causing extremely good or bad outcomes in all the states.

The rest of this section is devoted to A.2. Maccheroni, Marinacci and Rustichini (2006) introduce the weak certainty independence axiom which weakens the certainty independence axiom of Gilboa and Schmeidler (1989). Our A.2 is a further weakening of the weak certainty independence.

**A.2.1. Certainty Independence.** For all $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0, 1)$,

$$f \gtrsim g \Leftrightarrow \alpha f + (1-\alpha)x \gtrsim \alpha g + (1-\alpha)x.$$

**A.2.2. Weak Certainty Independence.** For all $f, g \in \mathcal{F}$, $x, y \in X$ and $\alpha \in (0, 1)$,

$$\begin{aligned} &\alpha f + (1-\alpha)x \gtrsim \alpha g + (1-\alpha)x \\ &\qquad \Rightarrow \alpha f + (1-\alpha)y \gtrsim \alpha g + (1-\alpha)y. \end{aligned} \tag{2.4}$$

Maccheroni, Marinacci and Rustichini (2006) show that a preference $\gtrsim$ satisfies A.2.1 if and only if for all $f, g \in \mathcal{F}$, $x, y \in X$ and $\alpha, \beta \in (0, 1]$,

$$\begin{aligned} &\alpha f + (1-\alpha)x \gtrsim \alpha g + (1-\alpha)x \\ &\qquad \Rightarrow \beta f + (1-\beta)y \gtrsim \beta g + (1-\beta)y. \end{aligned}$$

Thus A.2.2 weakens A.2.1 to require that the preference of two acts is only independent of the constant acts that they are mixed with, but not the weights in the mixing. Grant and Polak (2011) show that under the previous axioms, A.2.2 is equivalent to their constant absolute uncertainty aversion axiom which assumes the same condition (2.4) to hold only when $g$ is constant.[6]

**A.2.3. Constant Absolute Uncertainty Aversion.** For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)x \gtrsim \alpha z + (1-\alpha)x \\ \Rightarrow \alpha f + (1-\alpha)y \gtrsim \alpha z + (1-\alpha)y. \tag{2.5}$$

Our Axiom A.2 naturally extends A.2.3 to decreasing absolute uncertainty aversion by assuming (2.5) to hold either when $f$ is constant or $y \gtrsim x$. In this way, it differentiates the effect of changing a certainty part on constant acts and that on non-constant acts.

First suppose that $f$ is constant. Then A.2 is essentially von Neumann-Morgenstern's independence axiom on constant acts. It is the key to get an affine utility function, say $u$, to represent a preference on constant acts. Hence, changing a certainty part in *any* constant act, say from $x$ to $y$ by $1-\alpha$ proportion, the change in its utility is $(1-\alpha)(u(y)-u(x))$. In this sense, changing a certainty part generates the *same* effect on all constant acts. Thus, the preference is preserved under such a change.

Second, if $f$ is not constant, then the preference is preserved only when $y \gtrsim x$. While the effect of increasing a certainty part on constant acts can be normalized by the analysis above, (2.4) means that increasing a certainty part creates weakly larger improvement on a non-constant act than on a constant act. Equivalently, if the uncertainty of a non-constant act is tolerable when compared to a constant act, then it is even more tolerable as the certainty part grows.

The following example is a variation on Ellsberg (1961)'s thought experiment. It shows different behavioral implications of A.2 and the three axioms above.

[6] Actually Grant and Polak (2011) show the equivalence under A.1, A.3, a weaker version of A.4 and A.6.

**Example 1.** *An urn contains 100 balls, of which* 33 *are red, and* 67 *are either black or yellow. A ball is drawn from the urn. For each* $t \geq 0$, $r_t$ *denotes the act "betting on red". It pays* $100 + t$ *dollars if the ball is red and t dollars otherwise. Let* $b_t$ *denote the act "betting on black", and its payoff is analogously given. See the table below.*

Table 2.1 : Payoffs of $r_t$ and $b_t$

| $t \geq 0$ | Red | Black | Yellow |
|---|---|---|---|
| $r_t$ | 100+t | t | t |
| $b_t$ | t | 100+t | t |

*For instance,* $t = 0$ *and* $t = 10^4$.

Table 2.2 : Payoffs of $r_0$ and $b_0$

| $t = 0$ | Red | Black | Yellow |
|---|---|---|---|
| $r_0$ | 100 | 0 | 0 |
| $b_0$ | 0 | 100 | 0 |

Table 2.3 : Payoffs of $r_{10^4}$ and $b_{10^4}$

| $t = 10^4$ | Red | Black | Yellow |
|---|---|---|---|
| $r_{10^4}$ | 10,100 | 10,000 | 10,000 |
| $b_{10^4}$ | 10,000 | 10,100 | 10,000 |

*Suppose that the decision maker's preference* $\succsim$ *satisfies A.1, A.3, A.4 and A.6, and assume for simplicity that he is risk neutral. Then A.2.1, A.2.2 and A.2.3 each implies that*

*either* $r_t \succsim b_t$ *for all t, or* $b_t \succsim r_t$ *for all t.*

*However, our A.2 allows the existence of a threshold $\bar{t}$ such that*

$$r_t \succsim b_t \text{ for all } t \leq \bar{t}, \text{ and } b_t \succsim r_t \text{ for all } t \geq \bar{t}.$$

*Hence, A.2 accommodates the phenomenon that people become more willing to take uncertainty-bearing behavior as the baseline wealth increases.*

## 2.4 Representations

Let $u : X \to \mathbb{R}$ be a utility function of outcomes. Given $f \in \mathcal{F}$, let $u(f)$ denote a function in $\mathbb{R}^S$ assigning $u(f(s))$ to each $s \in S$. Thus $u(f)$ transfers each act $f$ to a state-contingent utility function. Let $I : u(X)^S \to \mathbb{R}$ be a functional on all state-contingent utility functions. We say that $I$ is weakly increasing if $I(\varphi) \geq I(\psi)$ whenever $\varphi, \psi \in u(X)^S$ and $\varphi(s) \geq \psi(s)$ for all $s \in S$. Let $\mathbf{1}$ denote a function in $\mathbb{R}^S$ such that $\mathbf{1}(s) = 1$ for all $s \in S$. Similarly, for any $t \in \mathbb{R}$, $t\mathbf{1}$ denotes a function in $\mathbb{R}^S$ that gives $t$ to each state $s \in S$. Define $\mathbb{R}\mathbf{1} = \{t\mathbf{1} \in \mathbb{R}^S | t \in \mathbb{R}\}$ to be the set of all constant utility functions on $S$.

Given $\varphi \in \mathbb{R}^S$, $p \in \Delta$, we denote by $E_p\varphi$ the expected value of $\varphi$ with respect to $p$. Let $d(p, q)$ denote the Euclidean distance between two priors $p, q \in \Delta$, $d(p, A)$ that between a prior $p \in \Delta$ and a set $A \subseteq \Delta$, and $d(A, B)$ that between two sets $A, B \subseteq \Delta$. Lastly, we equip the space $\mathbb{R}^S$ with the topology induced by the supremum norm.

### 2.4.1 Variant constraint representation

**Definition 9.** *A variant constraint representation of a preference $\succsim$ is a triple $\langle u, B, \eta \rangle$ such that*

*(1) $u : X \to \mathbb{R}$ is a non-constant affine utility function, $B$ is a non-empty closed convex subset of $\Delta$, and $\eta : u(X)^S \to \mathbb{R}_+$ is a distance constraint function;*

*(2) for* $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{p \in \{p \in \Delta | d(p,B) \leq \eta(\varphi)\}} E_p\varphi, \ \forall \varphi \in u(X)^S, \tag{2.6}$$

*I is weakly increasing and quasi-concave;*

*(3) for* $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)), \ \forall f \in \mathcal{F}, \tag{2.7}$$

*V represents* $\gtrsim$ *.*

The interpretation is that the decision maker considers each act as a state-contingent utility function. He has a set $B$ of approximating or best-guess priors, but he does not fully trust them. To make decisions robust to prior misestimation, he evaluates an act by the minimum expected utility over all the priors within $\eta$ distance of the approximating ones. The distance constraint $\eta$ measures the degree of his concern for prior misestimation, and it is a function of utility profiles.

**Theorem 5.** *The following statements are equivalent.*

*(1) A preference* $\gtrsim$ *satisfies A.1 - A.6.*

*(2) There exist an affine onto function* $u : X \to \mathbb{R}$*, a non-empty closed convex set* $B \subseteq \Delta$ *and a function* $\eta : \mathbb{R}^S \to \mathbb{R}_+$ *such that*

*(i)* $\langle u, B, \eta \rangle$ *is a variant constraint representation of* $\gtrsim$ *;*

*(ii)* $\eta$ *is continuous on* $\mathbb{R}^S \setminus \mathbb{R}\mathbf{1}$*,* $\eta(\varphi + t\mathbf{1})$ *weakly decreases in t for all* $\varphi \in \mathbb{R}^S$*, and* $\lim_{k \searrow 0} \lim_{t \to \infty} \eta(k\varphi + t\mathbf{1}) = 0.$

*Moreover, if* $\langle u', B', \eta' \rangle$ *also satisfies the conditions in (2), then* $u' = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*,* $B' = B$*, and* $\eta'(a\varphi + b) = \eta(\varphi)$ *for all* $\varphi \in \mathbb{R}^S$ *such that* $I(\varphi) \neq \min_{s \in S} \varphi(s)$ *where I is given by* $\langle u, B, \eta \rangle$ *as in* (2.6).

The distance constraint function $\eta$ has three properties. First, $\eta$ is continuous at every non-constant $\varphi \in \mathbb{R}^S$. When $\varphi$ is constant, the value of $I(\varphi)$ is identical for any distance constraint. However, $I$ is still continuous on the whole domain $\mathbb{R}^S$ of state-contingent utility functions.

The second property of $\eta$ is exactly the result of A.2. As the certainty utility increases, the decision maker may become more tolerant to prior misestimation, and thus reduces the range of priors under consideration. This implies that for $I$ in (2.6), $I(\varphi + t\mathbf{1}) \geq I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \geq 0$. Clearly, the equality holds when $\varphi$ is constant. This shows that increasing the baseline utility generates weakly more improvement on a non-constant act than on a constant one.

The third property of $\eta$ reveals the decision maker's uncertainty aversion in the limiting case. It says that the decision maker tends not to consider any prior outside the best-guess set in the "extremely good" situation where first the baseline utility increases to $\infty$, and second the scale of the uncertain part diminishes to 0. To understand the latter, see the following example. It is a variation on Maccheroni, Marinacci and Rustichini (2006)'s Example 2.

**Example 2.** *An urn contains 100 balls, of which 33 are red, and 67 are either black or yellow. A ball is drawn from the urn. For each $t > 0$, the act $r_t$, betting on red, pays $t$ dollars if the ball is red, and $t$ cents otherwise. The act $b_t$, betting on black, is defined analogously. See the following table of payoffs.*

Table 2.4 : Payoffs of $r_t$ and $b_t$

| $t > 0$ | Red | Black | Yellow |
|---|---|---|---|
| $r_t$ | t | 0.01t | 0.01t |
| $b_t$ | 0.01t | t | 0.01t |

*For example,* $t = 10$ *and* $t = 10^4$.

Table 2.5 : Payoffs of $r_{10}$ and $b_{10}$

| $t = 10$ | Red | Black | Yellow |
|---|---|---|---|
| $r_{10}$ | 10 | 0.1 | 0.1 |
| $b_{10}$ | 0.1 | 10 | 0.1 |

Table 2.6 : Payoffs of $r_{10^4}$ and $b_{10^4}$

| $t = 10^4$ | Red | Black | Yellow |
|---|---|---|---|
| $r_{10^4}$ | 10,000 | 100 | 100 |
| $b_{10^4}$ | 100 | 10,000 | 100 |

*The scale of money payment is measured by t. Assume for simplicity that the decision maker displays constant relative risk aversion* $\rho \in (0, 1)$*. Then t is basically the same as the utility scalar k in Theorem 1 (2/ii). The decision maker may become more willing to take the uncertainty-bearing behavior when the payoff scale t decreases. As a result, there may exist a threshold value* $\bar{t}$ *such that*

$$b_t \succsim r_t \text{ for all } t \leq \bar{t}, \text{ and } r_t \succsim b_t \text{ for all } t \geq \bar{t}.$$

*This can be the case when his preference satisfies A.1, A.3 - A.6, and A.2.2 — the weak certainty independence*[7]*. In this case,* $I(k\varphi) \leq kI(\varphi)$ *for all* $\varphi \in \mathbb{R}^S$ *and* $k \geq 1$*. Alternatively, in our representation, it means that* $\eta(\varphi) \leq \eta(k\varphi)$ *for all* $\varphi \in \mathbb{R}^S$ *such that* $I(\varphi) \neq \min_S \varphi(s)$ *and* $k \geq 1$*. If* $I(\varphi) = \min_S \varphi(s)$*, then* $\eta(\varphi)$ *and* $\eta(k\varphi)$ *can take any value which is above some lower bound. This property is referred as increasing relative uncertainty aversion*

[7] See Maccheroni, Marinacci and Rustichini (2006).

*(see e.g. Strzalecki (in press), Chateauneuf and Faro (2009); see also Section 5.2 for more discussion). It is basically the result of A.2.2 and A.5. If A.2.2 is weakened to A.2, then this property holds only when the certainty utility decreases to* $-\infty$*. More precisely,* $\lim_{t\to\infty}\eta(\varphi - t\boldsymbol{1}) \leq \lim_{t\to\infty}\eta(k\varphi - t\boldsymbol{1})$ *for all* $\varphi \in \mathbb{R}^S$ *such that* $I(\varphi) \neq \min_S \varphi(s)$ *and* $k \geq 1$*. Intuitively, it says that the payoff scale becomes an issue when the baseline utility is sufficiently low.*

### 2.4.2 Weighted maxmin representation

**Definition 10.** *A* weighted maxmin representation *of a preference* $\succsim$ *is a triple* $\langle u, C, \lambda\rangle$ *such that*

*(1)* $u : X \to \mathbb{R}$ *is a non-constant affine utility function,* $C$ *is a non-empty closed convex subset of* $\Delta$*, and* $\lambda : u(X)^S \to [0, 1]$ *is a weight function;*

*(2) for* $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \lambda(\varphi) \min_{p\in C} E_p\varphi + (1 - \lambda(\varphi)) \max_{p\in C} E_p\varphi \;\forall\varphi \in u(X)^S, \tag{2.8}$$

*$I$ is weakly increasing and quasi-concave;*

*(3) for* $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)) \;\forall f \in \mathcal{F},$$

*$V$ represents* $\succsim$*.*

The interpretation is that the decision maker considers each act $f$ as a utility profile $u(f)$. He believes that $C$ is the set of all possible priors, and evaluates an act by a weighted average of the best and worst expected utility over the priors in $C$. The weight $\lambda$ that he puts on the worst case is a function of utility profiles.

**Theorem 6.** *The following statements are equivalent.*

*(1) A preference* $\succsim$ *satisfies A.1 - A.6.*

*(2) There exist an affine onto function* $u : X \to \mathbb{R}$*, a non-empty closed convex set* $C \subseteq \Delta$ *and a function* $\lambda : \mathbb{R}^S \to [0, 1]$ *such that*

*(i)* $\langle u, C, \lambda \rangle$ *is a weighted maxmin representation of* $\succsim$ *;*

*(ii)* $\lambda$ *is continuous on* $\{\varphi \in \mathbb{R}^S | \min_{p \in C} E_p \varphi < \max_{p \in C} E_p \varphi\}$*,* $\lambda(\varphi + t\mathbf{1})$ *weakly decreases in* $t$ *for all* $\varphi \in \mathbb{R}^S$*, and* $\lim_{k \to \infty} \lim_{t \to \infty} \lambda(k\varphi - t\mathbf{1}) = 1$.

*Moreover, if* $\langle u', C', \lambda' \rangle$ *also satisfies the conditions in (2), then* $u' = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*,* $C' = C$*, and* $\lambda'(a\varphi + b) = \lambda(\varphi)$ *for all* $\varphi \in \mathbb{R}^S$ *such that* $\min_{p \in C} E_p \varphi < \max_{p \in C} E_p \varphi$.

The properties of the weight function $\lambda$ are analogous to those of $\eta$. The limiting condition of $\lambda$ says that the decision maker exhibits significant uncertainty aversion in the "extremely bad" situation, so that he tends to consider only the worst case. The "extremely bad" situation means that first, the baseline utility drops to $-\infty$, and second, the scale of the uncertain part expands to $\infty$.

Next we give a short discussion about the relationship between the variant constraint representation in Theorem 1 and the weighted maxmin representation in Theorem 2. In fact, they are dual to each other in the sense that they respectively describe a decision maker's behavior in the "extremely good" situation and in the "extremely bad" situation. The set $B$ in Theorem 1 reveals the priors that the decision maker always consider in minimizing expected utilities. He would never be "bold" enough to ignore any of them in any situation. The set $C$ in Theorem 2 reveals the priors that the decision maker would minimize over when evaluating some act.

**Corollary 1.** *Let* $\langle u, B, \eta \rangle$ *be the variant constraint representation as in Theorem 1 and* $\langle u, C, \lambda \rangle$ *the weighted maxmin representation as in Theorem 2. Then* $B \subseteq C$.

These two sets respectively provide an upper and lower bound for the evaluation. The value of any act is below the worst expected utility over $B$ and above that over $C$. The two bounds are tight. At best, the decision maker only considers the priors in $B$; at worst, he considers all the priors in $C$.

### 2.4.3 DAUA variational representation

**Definition 11.** *A DAUA variational representation of a preference* $\succsim$ *is a pair* $< u, c >$ *such that*

*(1)* $u : X \to \mathbb{R}$ *is a non-constant affine utility function, and* $c : u(X) \times \Delta \to [0, \infty]$ *is a lower semicontinuous cost function satisfying that (i) the function* $c(t, p) + t : u(X) \times \Delta \to (-\infty, \infty]$ *is quasi-convex, (ii)* $c(t, p)$ *is weakly increasing in* $t$ *for each* $p \in \Delta$*, and (iii)* $\inf_{p \in \Delta} c(t, p) = 0$ *for each* $t \in u(X)$*;*

*(2) for* $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{p \in \Delta}[E_p\varphi + c(E_p\varphi, p)], \ \forall \varphi \in u(X)^S, \tag{2.9}$$

*I is continuous;*

*(3) for* $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)), \ \forall f \in \mathcal{F},$$

*V represents* $\succsim$*.*

The DAUA variational representation generalizes Maccheroni, Marinacci and Rustichini (2006)'s variational representation.

**Definition 12.** *A variational representation of a preference* $\succsim$ *is a pair* $\langle u, c \rangle$ *such that*

*(1)* $u : X \to \mathbb{R}$ *is a non-constant affine utility function, and* $c : \Delta \to [0, \infty]$ *is a lower semicontinuous convex cost function satisfying* $\inf_{p \in \Delta} c(p) = 0$*;*

*(2) for $I : u(X)^S \to \mathbb{R}$ defined as*

$$I(\varphi) = \min_{\Delta}[E_p\varphi + c(p)],\ \forall\varphi \in u(X)^S,$$

*and for $V : \mathcal{F} \to \mathbb{R}$ defined as*

$$V(f) = I(u(f)),\ \forall f \in \mathcal{F},$$

*$V$ represents $\succsim$.*

*A preference $\succsim$ admitting a variational representation is called a variational preference.*

The DAUA variational representation generalizes the cost function in the variational representation by allowing a prior to have different costs at different utility levels. In particular, the cost of a prior weakly increases in utility, which corresponds to the DAUA axiom. Indeed, when the certainty utility of an act increases, say by $t$, the expected utility with respect to every prior is increased by $t$. Moreover, the cost of each prior is measured at a higher utility level and thus also weakly increases. As a result, the value of the act is increased by at least $t$, or equivalently, the act is evaluated by a more “favorable" prior. This means that decision maker becomes less averse to uncertainty when he becomes better off overall. With the cost function being constant in the utility term, the variational representation implies constant absolute uncertainty aversion.

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) obtain a general representation for a class of uncertainty averse preferences. The DAUA variational preferences include variational preferences, and are contained in this class of uncertainty averse preferences. By restricting the result of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) to the preferences displaying decreasing absolute uncertainty aversion, we obtain the following representation.

**Proposition 2.** *The following statements are equivalent.*

*(1) A preference* $\succsim$ *satisfies A.1 - A.6.*

*(2) There exist an affine onto function* $u : X \to \mathbb{R}$ *and a function* $c : \mathbb{R} \times \Delta \to [0, \infty]$ *such that* $\langle u, c \rangle$ *is a DAUA variational representation of* $\succsim$.

*Moreover, u is unique up to a positive affine transformation. For each u, c is uniquely given by* $c(t, p) = \sup\{u(x_f) - t | E_p u(f) \leq t, f \in \mathcal{F}\}$ *where* $(t, p) \in \mathbb{R} \times \Delta$.

*Lastly, let B and C be given as in Theorem 1 and Theorem 2 respectively. Then* $B = \{p \in \Delta | c(t, p) = 0$ *for all* $t \in \mathbb{R}\}$*, and* $C = \{p \in \Delta | c(t, p) < \infty\}$ *for any* $t \in \mathbb{R}$.

This representation is related to the previous two via the cost function. The set of priors with zero cost at any utility level coincides with the set $B$ of all best-guess priors. Both sets reveal the priors which are always considered in the evaluation.

Besides, for any utility level $t \in \mathbb{R}$, the set of priors that have finite cost at $t$ coincides with $C$. This means two things. First, if a prior has a finite cost at some utility level, then it has a finite cost at all utility levels. This property is not obvious. It basically comes from the interaction of A.2, A.5 and A.6. Second, the set of all priors that have finite cost at some utility level coincides with $C$. They reveal the priors that the decision maker would consider in some situation.

It follows that if a prior has an infinite cost at some utility level, then it has an infinite cost at all utility levels. The set of priors that have infinite cost at all utility levels is exactly $\Delta \setminus C$. Those priors are always excluded from consideration. No act is evaluated below the expected utility over $C$.

## 2.5 Special cases

### 2.5.1 Variational preferences

The class of variational preferences introduced by Maccheroni, Marrinacci and Rustichini (2006) is a subclass of DAUA variational preferences. They satisfy the stronger A.2.2 and thus display constant absolute uncertainty aversion. When restricting the previous results to this subclass, we again get three equivalent representations. The last representation is reduced to the variational representation. Hence, we find two other equivalent representations for variational preferences.

**Proposition 3.** *Suppose that* $\succsim$ *is a DAUA variational preference. Let* $\langle u, B, \eta\rangle$ *be the variant constraint representation as in Theorem 1,* $\langle u, C, \lambda\rangle$ *the weighted maxmin representation as in Theorem 2, and* $\langle u, c\rangle$ *the DAUA variational representation as in Proposition 1. Then the following statements are equivalent.*

*(1) The preference* $\succsim$ *satisfies A.2.2.*

*(2) For the distance constraint function* $\eta$, $\eta(\varphi + t\mathbf{1}) = \eta(\varphi)$ *for all* $t \in \mathbb{R}$ *and* $\varphi \in \mathbb{R}^S$.

*(3) For the weight function* $\lambda$, $\lambda(\varphi + t\mathbf{1}) = \lambda(\varphi)$ *for all* $t \in \mathbb{R}$ *and* $\varphi \in \mathbb{R}^S$.

*(4) For the cost function c,* $c(t, p) = c(t', p)$ *for all* $t, t' \in \mathbb{R}$ *and* $p \in \Delta$.

When the three representations are restricted to variational preferences, the distance constraint function $\eta$ and the weight function $\lambda$ do not change with the baseline utility, and the cost function $c$ does not depend on the utility term. This implies for the functional form $I$ in either (2.6), (2.8) or (2.9) that $I(\varphi + t\mathbf{1}) = I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \in \mathbb{R}$. It means that adding $t$ units of utility in each state for any utility profile $\varphi$, the change of the value is always $t$, which is independent of $\varphi$. Thus, changing the certainty part generates the same effect on all acts. Hence, the degree of the decision maker's uncertainty aversion is fixed.

Proposition 2 shows that the variant constraint representation satisfying condition (2) and the weighted maxmin representation satisfying condition (3) are equivalent to the variational representation. The two equivalent representations provide different perspectives on variational preferences which are not obvious from the variational representation.

**Corollary 2.** *Suppose that* $\succsim$ *is a variational preference. Let* $\langle u, B, \eta\rangle$*,* $\langle u, C, \lambda\rangle$ *and* $\langle u, c\rangle$ *be the three equivalent representations for* $\succsim$ *as in Proposition 2. Then the following statements hold.*

*(1) For the distance constraint function* $\eta$*,* $\eta(k\varphi) \geq \eta(\varphi)$ *for all* $k \geq 1$ *and* $\varphi \in \mathbb{R}^S$ *such that* $I(\varphi) \neq \min_{s\in S} \varphi(s)$ *where* $I$ *is given as in (2.6), and* $\lim_{k\searrow 0} \eta(k\varphi) = 0$ *for all* $\varphi \in \mathbb{R}^S$*.*

*(2) For the weight function* $\lambda$*,* $\lambda(k\varphi) \geq \lambda(\varphi)$ *for all* $k \geq 1$ *and* $\varphi \in \mathbb{R}^S$*, and* $\lim_{k\to\infty} \lambda(k\varphi) = 1$ *for all* $\varphi \in \mathbb{R}^S$*.*

*(3) The set* $B = \{p \in \Delta | c(p) = 0\}$ *and the set* $C = \{p \in \Delta | c(p) < \infty\}$*.*

The first two representations directly show that variational preferences exhibit relative increasing uncertainty aversion. This property has been discussed in Section 4.1. It mainly comes from the interaction of A.2.2 and A.5. As a result, when the size of $\varphi$ is scaled up, the decision maker becomes weakly more averse to uncertainty. Thus he considers more priors which are farther away from the best-guess ones, or he puts more weight on the worst case. Moreover, as the wealth effect is assumed away, only the scale effect plays a role in the limiting conditions. When the scalar $k$ of a utility profile diminishes to 0, then the decision maker tends to evaluate an act by the worst expected utility over only the priors in $B$. When $k$ expands to $\infty$, he tends to consider all the priors in $C$.

Lastly, the set of zero-cost priors and the set of finite-cost priors in the variational representation coincide respectively with $B$ and $C$. This provides a further understanding of the cost function $c$.

### 2.5.2 Maxmin preferences

**Definition 13.** *A maxmin representation of a preference $\succsim$ is a pair $\langle u, C\rangle$ such that*

*(1) $u : X \to \mathbb{R}$ is a non-constant affine utility function, and $C$ is a non-empty closed convex subset of $\Delta$;*

*(2) for $V : \mathcal{F} \to \mathbb{R}$ defined as*

$$V(f) = \min_{p \in C} E_p u(f),\ \forall f \in \mathcal{F},$$

*$V$ represents $\succsim$.*

*A preference $\succsim$ admitting a maxmin representation is called a maxmin preference.*

Gilboa and Schmeidler (1989)'s maxmin preferences are special cases of variational preferences. Maxmin preferences satisfy the stronger A.2.1. They display not only constant absolute uncertainty aversion, but also constant relative uncertainty aversion.

The maxmin representation can be viewed as a variational representation with the cost function satisfying that $c(p) = 0$ for all $p \in C$ and $c(p) = \infty$ otherwise. The following result shows how the maxmin model fits in our representations.

**Proposition 4.** *Suppose that $\succsim$ is a DAUA variational preference. Let $\langle u, B, \eta\rangle$ be the variant constraint representation as in Theorem 1, $\langle u, C, \lambda\rangle$ the weighted maxmin representation as in Theorem 2, and $\langle u, c\rangle$ the DAUA variational representation as in Proposition 1. Then the following statements are equivalent.*

*(1) The preference $\succsim$ satisfies A.2.1.*

*(2) For the distance constraint function $\eta$, $\eta(\varphi) = 0$ for all $\varphi \in \mathbb{R}^S$ such that $I(\varphi) \neq \min_{s \in S} \varphi(s)$ where $I$ is given as in (2.6).*

*(3) For the weight function $\lambda$, $\lambda(\varphi) = 1$ for all $\varphi \in \mathbb{R}^S$.*

*(4) The set $B$ coincides with the set $C$.*

*(5) For the cost function c,* $c(t, p) = 0$ *for all* $t \in \mathbb{R}$ *and* $p \in B$*, and* $c(t, p) = \infty$ *for all* $t \in \mathbb{R}$ *and* $p \in \Delta \setminus B$*.*

If a decision maker has a maxmin preference, then the set of best-guess priors is the set of all possible priors. He always evaluates an act by the worst expected utility over this set. Moreover, at any utility level, the priors with finite cost are exactly those with zero cost. The equivalence between (1) and (5) is essentially Proposition 19 of Maccheroni, Marinacci and Rustchini (2006).

### 2.5.3 Constraint preferences and multiplier preferences

Hansen and Sargent (2001) introduce two robust decision rules to model the situation where a decision maker facing uncertainty has an approximating probabilistic model and is also concerned about prior misestimation. The two rules are the constraint criterion and the multiplier criterion. Before giving their formulas, we first introduce the definition of relative entropy.

Given $p, q \in \Delta$, $p \ll q$ denotes that $p$ is absolutely continuous with respect to $q$.

**Definition 14.** *Given* $p, q \in \Delta$*, the relative entropy* $R(p\|q)$ *of* $p$ *with respect to* $q$ *is defined by*

$$R(p\|q) = \begin{cases} \sum_S p_i \log \dfrac{p_i}{q_i} & \text{if } p \ll q \\ \infty & \text{otherwise.} \end{cases}$$

Relative entropy is a measure of the “difference" or “distance" between two probabilities. Note that $R(\cdot\|q) : \Delta \to [0, \infty]$ is a lower semicontinuous convex function satisfying that $R(p\|q) = 0$ if and only if $p = q$.

**Definition 15.** *A constraint representation of a preference* $\succsim$ *is a triple* $\langle u, q, \eta \rangle$ *such that*

*(1) $u : X \to \mathbb{R}$ is a non-constant affine utility function, $q \in \Delta$ is an approximating prior, and $\eta \in [0, \infty)$ is a parameter;*

*(2) for $V : \mathcal{F} \to \mathbb{R}$ defined as*

$$V(f) = \min_{p \in \{p \in \Delta | R(p\|q) \leq \eta\}} E_p u(f), \ \forall f \in \mathcal{F}, \tag{2.10}$$

*$V$ represents $\succsim$.*

*A preference $\succsim$ admitting a constraint representation is called a constraint preference.*

Constraint representation differs from variant constraint representation in three respects. First, the decision maker has a single best-guess prior $q$ instead of a set of them. Second, the “distance" between two priors is measured by relative entropy instead of the Euclidean metric. Third, the distance constraint $\eta$ is a constant rather than a function of utility profiles. The parameter $\eta$ measures the degree of his concern with prior misspecification. Larger values of $\eta$ correspond to less trust in $q$.

Because of the properties of relative entropy, $\{p \in \Delta | R(p\|q) \leq \eta\}$ is a non-empty closed convex set. Thus, constraint preferences are maxmin preferences with the constraint set being specified in a concrete way.

**Definition 16.** *A multiplier representation of a preference $\succsim$ is a triple $\langle u, q, \theta \rangle$ such that*

*(1) $u : X \to \mathbb{R}$ is a non-constant affine utility function, $q \in \Delta$ is an approximating prior, and $\theta \in [0, \infty)$ is a parameter;*

*(2) for $V : \mathcal{F} \to \mathbb{R}$ defined as*

$$V(f) = \min_{p \in \Delta} [E_p(u(f)) + \theta R(p\|q)], \ \forall f \in \mathcal{F}, \tag{2.11}$$

*$V$ represents $\succsim$.*

*A preference $\succsim$ admitting a multiplier representation is called a multiplier preference.*

The multiplier representation is a special variational representation with the cost function being $\theta R(p||q)$. A decision maker with a multiplier preference has an approximating prior $q$. He considers each prior $p$, but this incurs a cost which is proportional to the “distance" between $p$ and $q$. The parameter $\theta$ measures the degree of his concern with prior misspecification. In contrast to $\eta$ in (2.10), Larger values of $\theta$ correspond to more trust in $q$. If $\theta = \infty$, then with the convention that $0 \cdot \infty = 0$, the decision maker evaluates each act by the expected utility with respect to $q$.

Hansen and Sargent (2001) establish the connection between constraint and multiplier criteria. They show in a dynamic resource allocation problem that under some conditions, for each $\eta$ in (2.10), there exists a $\theta$ in (2.11) such that (2.11) implies the same optimal solution as (2.10), and vise versa. However, as preference orderings, constraint criterion and multiplier criterion give totally different rankings of acts other than the optimal one. We obtain a constraint type of representation for multiplier preferences, which clearly shows the relationship between the two criteria.

Given $f, g \in \mathcal{F}$ and $A \subseteq S$, define $f_A g$ to be an act in $\mathcal{F}$ such that $f_A g(s) = f(s)$ for all $s \in A$, and $f_A g(s) = g(s)$ for all $s \in S \setminus A$. An event $A \subseteq S$ is *nonnull* if there exist $f, g, h \in \mathcal{F}$ such that $f_A h \succ g_A h$.

**Proposition 5.** *Suppose that $S$ has at least three nonnull events. Then the following two statements are equivalent.*

*(1) A preference $\succsim$ admits a multiplier representation $\langle u, q, \theta \rangle$.*

*(2) Define $V : \mathcal{F} \to \mathbb{R}$ by*

$$V(f) = \min_{p \in \{p \in \Delta | R(p||q) \leq \eta(u(f))\}} E_p u(f), \ \forall f \in \mathcal{F},$$

*where $\eta : u(X)^S \to \mathbb{R}_+$ is given by $\eta(u(f)) = \min\limits_{p \in \{p \in \Delta | E_p u(f) = -\theta \log E_q e^{-\frac{u(f)}{\theta}}\}} R(p||q)$ for all $f \in \mathcal{F}$.*

*The function V represents* $\succsim$.

This result provides an equivalent representation for multiplier preferences which can be directly compared with constraint representation. It shows that a multiplier preference also admit a “constraint representation" except that the “distance" constraint is a specific function of utility profiles rather than a fixed number. Thus, the multiplier criterion induces the changing robustness concern for different acts, while constraint criterion assumes the constant robustness concern for all acts.

In particular, the function $\eta$ has three properties. Suppose without loss of generality that $u(X) = \mathbb{R}$. First, $\eta(\phi + t\mathbf{1}) = \eta(\phi)$ for all $\phi \in \mathbb{R}^S$ and $t \in \mathbb{R}$, which shows that there is no wealth effect for multiplier preferences. Second, $\eta(k\phi) \geq \eta(\phi)$ if $k \geq 1$, which shows that multiplier preferences display increasing relative uncertainty aversion property. Lastly, $\lim_{k \searrow 0} \eta(k\phi) = 0$. This means that when the utility scale diminishes to 0, the decision maker tends to consider only his best-guess prior $q$. Moreover, the later two properties in turn imply that a multiplier preference and a constraint preference coincide if and only if both are represented by the subjective expected utility $V(f) = E_q u(f)$ for all $f \in \mathcal{F}$.

This equivalent representation of multiplier preferences closely resembles the variant constraint representation of DAUA variational preferences. The only difference lies in the measure of “distance" between priors. In fact, the Euclidean distance in the variant constraint representation can be replaced with a general measures of “distance". Let $l : \Delta \times \Delta \to \mathbb{R}_+$ be a lower semicontinuous function such that $l(\cdot, q) : \Delta \to \mathbb{R}_+$ is convex in the first term, and $l(p, q) = 0$ if and only if $p = q$. Define $L : \Delta \times \{B \subseteq \Delta | B \neq \emptyset\} \to \mathbb{R}_+$ by $L(p, B) = \inf_{q \in B} l(p, q)$ for any $p \in \Delta$ and non-empty set $B \subseteq \Delta$. Then all such functions $L$ can be used as a “distance" measure in the variant constraint representation for DAUA variational preferences which include multiplier preferences. These measures are equivalent in the sense that with the corresponding “distance" constraint functions $\eta$, they all induce the same evaluation of acts.

Although one can use the general “metric" as above to obtain another variant constraint representation for multiplier preferences, relative entropy cannot be used in general to derive the variant constraint representation of DAUA variational preferences. One feature of relative entropy is that it assigns $\infty$ to all the priors that are not absolutely continuous with respect to the central prior $q$. This implies that the decision maker believes for sure that the states to which $q$ assigns zero probability would never happen, and he disregards all the priors that assign positive probability to those states. This is not generally true for DAUA variational preferences. A decision maker with a DAUA variational preference typically considers all kinds of perturbation of the central prior $q$.

## 2.6 Conclusion

This paper axiomatizes a class of preferences which display decreasing absolute uncertainty aversion. We obtain three equivalent representations: variant constraint representation, weighted maxmin representation, and DAUA variational representation. This class of preferences includes variational preferences as a subclass. When restricted to this subclass, the first two representations are equivalent to the established variational representation. Moreover, a constraint type of representation is obtained for multiplier preferences. This representation directly shows the relationship between multiplier and constraint preferences.

In closing, we remark that three representations can be similarly obtained for the analogous class of preferences with *increasing* absolute uncertainty aversion. The only difference is that when the baseline utility of an act rises, the distance constraint $\eta$ and the weight function $\lambda$ weakly increase, while the cost function $c$ weakly decreases in utilities.

## 2.7 Appendix: proofs

We denote by $\mathbb{Z}_+$ the set of positive integers, $\Delta^\circ$ the interior of $\Delta$ and $\partial\Delta$ the boundary of $\Delta$. Let $I : \mathbb{R}^S \to \mathbb{R}$ be given. We say that $I$ is *normalized* if $I(t\mathbf{1}) = t$ for all $t \in \mathbb{R}$. We say that $I$ is *constant superadditive* if $I(\varphi + t\mathbf{1}) \geq I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \geq 0$. Similarly, $I$ is said to be *constant additive* if $I(\varphi + t\mathbf{1}) = I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \in \mathbb{R}$. If $I(\varphi + \varphi') \geq I(\varphi) + I(\varphi')$ for all $\varphi, \varphi' \in \mathbb{R}^S$, then $I$ is said to be *superadditive*.

**Lemma 7.** *A preference $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1 - A.6 if and only if there exists an affine onto function $u : X \to \mathbb{R}$ and a functional $I : \mathbb{R}^S \to \mathbb{R}$ such that*

*(1) it is normalized, weakly increasing, quasi-concave, continuous and constant superadditive;*

*(2) $f \succsim g \Leftrightarrow I(u(f)) \geq I(u(g))$ for all $f, g \in \mathcal{F}$.*

*Moreover, $u$ is unique up to a positive affine transformation, and given $u$, there is a unique normalized functional $I : \mathbb{R}^S \to \mathbb{R}$ such that the above condition (2) holds.*

*Proof.* The necessity is easy. For the sufficed, the existence and uniqueness of the required $u$ and $I$ follow from Maccheroni, Marinacci and Rustichini (2006) (Lemma 28), Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) (Lemma 57), and Kopylov (2001), except that $I$ is constant superadditive. We check now the constant superadditivity of $I$.

Let $\varphi \in \mathbb{R}^S$ and $t \geq 0$ be given. Suppose $x, x_0 \in X$ and $f \in \mathcal{F}$ such that $u(x) = 2t$, $u(x_0) = 0$ and $u(f) = 2\varphi$. Then $u(\frac{1}{2}f + \frac{1}{2}x) = \varphi + t\mathbf{1}$ and $u(\frac{1}{2}f + \frac{1}{2}x_0) = \varphi$. Since $u$ is an affine onto function, then there exists $z \in X$ such that $\frac{1}{2}f + \frac{1}{2}x_0 \sim \frac{1}{2}z + \frac{1}{2}x_0$ for some $z \in X$.

Since $t \geq 0$, then $x \gtrsim x_0$. By Axiom A.2, we know that $\frac{1}{2}f + \frac{1}{2}x \gtrsim \frac{1}{2}z + \frac{1}{2}x$. Thus,

$$\begin{aligned}
&I(\varphi + t\mathbf{1}) = I(u(\frac{1}{2}f + \frac{1}{2}x)) \geq I(u(\frac{1}{2}z + \frac{1}{2}x)) \\
&= \frac{1}{2}u(z) + \frac{1}{2}u(x) = \frac{1}{2}u(z) + \frac{1}{2}u(x_0) + \frac{1}{2}u(x) = u(\frac{1}{2}z + \frac{1}{2}x_0) + t \\
&= I(u(\frac{1}{2}f + \frac{1}{2}x_0)) + t = I(\varphi) + t.
\end{aligned}$$

Checking the equalities and inequalities above mainly use the fact that $I$ is normalized and satisfies condition (2), and that $u$ is affine. □

**Proof of Theorem 1.** We check the sufficiency first. Let $u : X \to \mathbb{R}$ and $I : \mathbb{R}^S \to \mathbb{R}$ be given as in Lemma 7. We define $J : \mathbb{R}^S \to \mathbb{R}$ by

$$J(\varphi) = \lim_{k \searrow 0} \lim_{t \to \infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] \tag{2.12}$$

for all $\varphi \in \mathbb{R}^S$.

To check that $J : \mathbb{R}^S \to \mathbb{R}$ is well defined, we show some stronger results.

First, for all $k > 0$ and $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$ is bounded in $[\min_S \varphi(s), \max_S \varphi(s)]$. This is because $I$ is weakly increasing.

Second, fix $k > 0$, $\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$ weakly increases in $t$. Indeed, if $t' \geq t$, then by the constant supadditivity of $I$, $I(k\varphi + t'\mathbf{1}) - t' = I(k\varphi + t\mathbf{1} + (t' - t)\mathbf{1}) - t' \geq I(k\varphi + t\mathbf{1}) + t' - t - t' = I(k\varphi + t\mathbf{1}) - t$.

Third, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$ weakly decreases in $k$ when $k > 0$. Suppose for the sake of a contradiction that $k' \geq k > 0$ and $\lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi + t\mathbf{1}) - t] > \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$. Hence, there exists $\bar{t}$ such that for all $t, t' \geq \bar{t}$, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] > \lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi + t\mathbf{1}) - t]$. That is,

$$I(k\varphi + t\mathbf{1}) < \frac{k}{k'} I(k'\varphi + t'\mathbf{1}) + t - \frac{k}{k'}t'. \tag{2.13}$$

Pick $t, t' \geq \bar{t}$ such that

$$\frac{k}{k'}t' + (1 - \frac{k}{k'})I(k'\varphi + t'\mathbf{1}) = t. \tag{2.14}$$

Thus, $k\varphi + t\mathbf{1} = \frac{k}{k'}(k'\varphi + t'\mathbf{1}) + (1 - \frac{k}{k'})I(k'\varphi + t'\mathbf{1})$. Since $I$ is normalized and quasi-concave, then

$$I(k\varphi + t\mathbf{1}) \geq \frac{k}{k'}I(k'\varphi + t'\mathbf{1}) = \frac{k}{k'}I(k'\varphi + t'\mathbf{1}) + t - \frac{k}{k'}t',$$

where the equality follows from the choice of $t$ and $t'$. This is a contradiction to (2.13) as desired.

The three results above guarantee that $J : \mathbb{R}^S \to \mathbb{R}$ is well defined. Note that $I(\varphi) \leq J(\varphi)$ for all $\varphi \in \mathbb{R}^S$.

We further show some properties of $J$.

It is easy to see that $J$ is normalized and weakly increasing. It directly follows from the same properties of $I$.

Besides, $J$ is constant additive. Let $\varphi \in \mathbb{R}^S$ and $r \in \mathbb{R}$ be given. Then

$$\begin{aligned} J(\varphi + r\mathbf{1}) &= \lim_{k\searrow 0}\lim_{t\to\infty}\frac{1}{k}[I(k(\varphi + r\mathbf{1}) + t\mathbf{1}) - t] \\ &= \lim_{k\searrow 0}\lim_{t\to\infty}\frac{1}{k}[I(k\varphi + (kr + t)\mathbf{1}) - (kr + t) + kr] \\ &= \lim_{k\searrow 0}\lim_{t\to\infty}\frac{1}{k}[I(k\varphi + (kr + t)\mathbf{1}) - (kr + t)] + r \\ &= \lim_{k\searrow 0}\lim_{t'\to\infty}\frac{1}{k}[I(k\varphi + t'\mathbf{1}) - t'] + r = J(\varphi) + r, \end{aligned}$$

as desired.

Moreover, $J$ is positive homogeneous of degree 1. Let $\varphi \in \mathbb{R}^S$ and $l > 0$ be given. Then

$$\begin{aligned}
J(l\varphi) &= \lim_{k\searrow 0}\lim_{t\to\infty}\frac{1}{k}[I(kl\varphi + t\mathbf{1}) - t] \\
&= \lim_{k\searrow 0}\lim_{t\to\infty}\frac{l}{kl}[I(kl\varphi + t\mathbf{1}) - t] = l\lim_{k\searrow 0}\lim_{t\to\infty}\frac{1}{kl}[I(kl\varphi + t\mathbf{1}) - t] \\
&= l\lim_{k'\searrow 0}\lim_{t\to\infty}\frac{1}{k'}[I(k'\varphi + t\mathbf{1}) - t] = lJ(\varphi),
\end{aligned}$$

as desired.

Lastly, $J$ is superadditive. Suppose for the sake of a contradiction that $J(\varphi + \varphi') < J(\varphi) + J(\varphi')$ for some $\varphi, \varphi' \in \mathbb{R}^S$. Since $J$ is positive homogeneous of degree 1, then $J(\frac{1}{2}\varphi + \frac{1}{2}\varphi') < \frac{1}{2}J(\varphi) + \frac{1}{2}J(\varphi')$. Thus, there exsit $k > 0$ and $\bar{t} \geq 0$ such that for all $t, t' \geq \bar{t}$ and $t'' \in \mathbb{R}$,

$$\frac{1}{k}[I(k(\frac{1}{2}\varphi + \frac{1}{2}\varphi') + t''\mathbf{1}) - t''] < \frac{1}{2k}[I(k\varphi + t\mathbf{1}) - t] + \frac{1}{2k}[I(k\varphi' + t'\mathbf{1}) - t']. \tag{2.15}$$

By rearranging the terms, we get

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' + t''\mathbf{1}) < \frac{1}{2}I(k\varphi + t\mathbf{1}) + \frac{1}{2}I(k\varphi' + t'\mathbf{1}) + t'' - \frac{t + t'}{2}. \tag{2.16}$$

Pick $t \geq \bar{t}$ such that $k\min_S \varphi(s) + t \geq k\max_S \varphi'(s) + \bar{t}$. Thus $I(k\varphi + t\mathbf{1}) \geq k\min_S \varphi(s) + t \geq k\max_S \varphi'(s) + \bar{t}$. Pick $t' \in \mathbb{R}$ such that $I(k\varphi + t\mathbf{1}) = I(k\varphi' + t'\mathbf{1})$. Thus $t' \geq \bar{t}$. Let $t'' = \frac{t+t'}{2}$ so that $\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' + t''\mathbf{1} = \frac{1}{2}(k\varphi + t\mathbf{1}) + \frac{1}{2}(k\varphi' + t'\mathbf{1})$. Since $I$ is superadditive, then

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' + t''\mathbf{1}) \geq \frac{1}{2}I(k\varphi + t\mathbf{1}) + \frac{1}{2}I(k\varphi' + t'\mathbf{1})$$

which is a contradiction to (2.16), as desired.

Since the functional $J : \mathbb{R}^S \to \mathbb{R}$ satisfies the properties above, then by Gilboa and

Schmeidler (1989)'s Lemma 3.5, there exists a non-empty closed convex set $B \subseteq \Delta$ such that for all $\varphi \in \mathbb{R}^S$, $J(\varphi) = \min_B E_p\varphi$.

Define a set $D(\varphi) \subseteq \{p \in \Delta | I(\varphi) = E_p\varphi\}$ for each $\varphi \in \mathbb{R}^S$. Note that $D(\varphi)$ is a non-empty compact set for all $\varphi \in \mathbb{R}^S$. Define $\eta : \mathbb{R}^S \to \mathbb{R}_+$ as $\eta(\varphi) = d(D(\varphi), B)$. We want to check that $I(\varphi) = \min_{\{p \in \Delta | d(p,B) \leq \eta(\varphi)\}} E_p\varphi$ for all $\varphi \in \mathbb{R}^S$.

Fix $\varphi \in \mathbb{R}^S$. Since $D(\varphi)$ is non-empty and compact, then by the definition of $\eta(\varphi)$, there exists $p_* \in D(\varphi)$ such that $d(p, B) = \eta(\varphi)$. Hence, $I(\varphi) \leq \min_{\{p \in \Delta | d(p,B) \leq \eta(\varphi)\}} E_p\varphi$. Now we check that for all $p \in \Delta$ with $d(p, B) \leq \eta(\varphi)$, $E_p\varphi \geq I(\varphi)$. Suppose for the sake of a contradiction that there exists $p' \in \Delta$ such that $d(p', B) \leq \eta(\varphi)$ and $E_{p'}\varphi < I(\varphi)$. Denote by $q$ a prior in $B$ such that $d(p', q) = d(p', B)$. Then $E_q\varphi \geq \min_B E_p\varphi \geq I(\varphi) > E_{p'}\varphi$. Thus there exists $\alpha \in [0, 1)$ such that $I(\varphi) = E_{\alpha p' + (1-\alpha)q}\varphi$, which means that $\alpha p' + (1 - \alpha)q \in D(\varphi)$. If $d(p', q) > 0$, then

$$\begin{aligned} & d(D(\varphi), B) \leq d(\alpha p' + (1 - \alpha)q, q) = \alpha d(p', q) \\ & < d(p', q) = d(p', B) \leq \eta(\varphi) \end{aligned}$$

which contradicts the definition of $\eta$. If $d(p', q) = 0$, then $p' = q$, and thus

$$E_q\varphi = E_{p'}\varphi < I(\varphi) \leq \min_B E_p\varphi$$

which is again a contradiction since $q \in B$.

Moreover, for all $p \in \Delta$ with $d(p, B) < \eta(\varphi)$, $E_p\varphi > I(\varphi)$. Indeed, if $d(p', B) < \eta(\varphi)$ and $E_{p'}\varphi = I(\varphi)$ for some $\varphi \in \mathbb{R}^S$ and $p' \in \Delta$, then $p' \in \mathbb{D}(\varphi)$. Thus, $\eta(\varphi) = \min_{D(\varphi)} d(p, B) \leq d(p', B) < \eta(\varphi)$, which is a contradiction.

Now we check the properties of $\eta : \mathbb{R}^S \to \mathbb{R}_+$.

First, $\eta$ is continuous on $\mathbb{R}^S \setminus \mathbb{R}\mathbf{1}$. Since $\eta(\varphi) = \min_{D(\varphi)} d(p, B)$ for all $\varphi \in \mathbb{R}^S$, by the maximum theorem, it suffices to check that $D : \mathbb{R}^S \rightrightarrows \Delta$ is continuous as a correspondence. Fix an arbitrary $\varphi \in \mathbb{R}^S$, and we check that $D$ is upper hemicontinuous at $\varphi$. Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim\limits_{n\to\infty} \varphi_n = \varphi$. Let $\{p_n\}_{n=1}^\infty$ be a sequence of elements in $\Delta$ such that $\lim\limits_{n\to\infty} p_n = p$ for some $p \in \Delta$. Then $E_p\varphi = \lim\limits_{n\to\infty} E_{p_n}\varphi_n = \lim\limits_{n\to\infty} I(\varphi_n) = I(\lim\limits_{n\to\infty} \varphi_n) = I(\varphi)$. Thus $p \in \mathbb{D}(\varphi)$.

Fix an arbitrary $\varphi \in \mathbb{R}^S \setminus \mathbb{R}\mathbf{1}$, and $D$ is lower hemicontinuous at $\varphi$. To see it, let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim\limits_{n\to\infty} \varphi_n = \varphi$. Fix $p \in D(\varphi)$. Suppose that $p \in \Delta^\circ$. Let $\epsilon > 0$ be arbitrarily given. Define $A(\epsilon) = \{q \in \Delta | d(q,p) \leq \epsilon\}$. Thus $E_p\varphi \in (\min\limits_{A(\epsilon)} E_q\varphi, \max\limits_{A(\epsilon)} E_q\varphi)$. There exists $N \in \mathbb{Z}_+$ such that $I(\varphi_n) \in (\min\limits_{A(\epsilon)} E_q\varphi_n, \max\limits_{A(\epsilon)} E_q\varphi_n)$ whenever $n \geq N$. Therefore, there exists $p_n \in A(\epsilon)$ for each $n \geq N$ such that $I(\varphi_n) = E_{p_n}\varphi_n$. Now for each $j \in \mathbb{Z}_+$, pick $p_{n_j} \in A(\frac{1}{j})$ such that $p_{n_j} \in D(\varphi_{n_j})$ and $n_{j'} > n_j$ for all $j' > j$ in $\mathbb{Z}_+$. Thus we get a sequence $\{p_{n_j}\}_{j=1}^\infty$ such that $p_{n_j} \in D(n_j)$ for all $j \in \mathbb{Z}_+$ and $\lim\limits_{j\to\infty} p_{n_j} = p$. Now if $p \in \partial\Delta$, one can find such a sequence via a sequence $\{p_m\}_{m=1}^\infty$ of elements in $\Delta^\circ$ that converges to $p$.

Second, $\eta(\varphi + t\mathbf{1})$ weakly decreases in $t \in \mathbb{R}$ for all $\varphi \in \mathbb{R}^S$. Let $\varphi \in \mathbb{R}^S$ and $t \leq t'$ in $\mathbb{R}$ be given. For all $p \in D(\varphi + t\mathbf{1})$, $E_p[\varphi + t'\mathbf{1}] = I(\varphi + t\mathbf{1}) + t' - t \leq I(\varphi + t'\mathbf{1})$. Suppose that $\eta(\varphi + t\mathbf{1}) = d(p', B)$ for some $p' \in \mathbb{D}(\varphi + t\mathbf{1})$. If $E_{p'}[\varphi + t'\mathbf{1}] < I(\varphi + t'\mathbf{1})$, then from the analysis above we know that $d(p', B) > \eta(\varphi + t'\mathbf{1})$, and thus $\eta(\varphi + t\mathbf{1}) > \eta(\varphi + t'\mathbf{1})$. If $E_{p'}[\varphi + t'\mathbf{1}] = I(\varphi + t'\mathbf{1})$, then $p' \in \mathbb{D}(\varphi + t'\mathbf{1})$, and thus $\eta(\varphi + t\mathbf{1}) = d(p', B) \geq \min\limits_{D(\varphi+t'\mathbf{1})} d(p, B) = \eta(\varphi + t'\mathbf{1})$.

Third, for any $\varphi \in \mathbb{R}^S$, $\lim\limits_{t\to\infty} \eta(k\varphi + t\mathbf{1})$ weakly increases in $k$ when $k > 0$. We first show that it is equivalent to check that $\min\limits_{\{q\in\Delta | \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]\}} d(q, B)$ weakly increased in $k$ when $k > 0$. Let $\varphi \in \mathbb{R}^S$ and $k > 0$ be given. Define $T : [0, \infty] \rightrightarrows \Delta$ as a correspondence by $T(t) = \{p \in \Delta | E_p\varphi = \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]\}$ for each $t \in \mathbb{R}$, and $T(\infty) = \{p \in \Delta | E_p\varphi = \lim\limits_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]\}$. Note that for all $t \in [0, \infty]$, $T(t)$ is non-empty and compact. We show first that $\min_{T(\infty)} d(p,B) = \lim_{t\to\infty} \theta(k\varphi + t\mathbf{1})$. To do it, we check that $T$ is continuous at $\infty$. For the upper hemicontinuity, let $\{t_n\}_{n=1}^{\infty}$ be a sequence of elements in $[0,\infty]$ such that $\lim_{n\to\infty} t_n = \infty$, and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements in $\Delta$ such that $p_n \in T(t_n)$ for all $n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} p_n = p$ for some $p \in \Delta$. Then $E_p\varphi = \lim_{n\to\infty} E_{p_n}\varphi = \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$. Hence, $p \in T(\infty)$. For the lower hemicontinuous, let $\{t_n\}_{n=1}^{\infty}$ be a sequence of elements in $[0,\infty]$ such that $\lim_{n\to\infty} t_n = \infty$. Fix $p \in T(\infty)$. Suppose that $q \in \Delta$ and $E_q\varphi = \min_S \varphi(s)$. If $E_p\varphi = E_q\varphi$, then $p \in T(t_n)$ for all $n \in \mathbb{Z}_+$ and thus we are done. Suppose that $E_p\varphi > E_q\varphi$. Then for each $n \in \mathbb{Z}_+$, there exists a unique $\alpha_n \in [0,1]$ such that $\alpha_n p + (1-\alpha_n)q \in T(t_n)$. Therefore, $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{1}{E_p\varphi - E_q\varphi}[\frac{1}{k}I((k\varphi + t_n\mathbf{1}) - t_n) - E_q\varphi] = \frac{1}{E_p\varphi - E_q\varphi}(E_p\varphi - E_q\varphi) = 1$. Thus, $\lim_{n\to\infty}[\lambda_n p + (1-\lambda_n)q] = p$. Hence, by the maximum theorem, $\min_{T(infty)} d(q,B) = \lim_{t\to\infty} \min_{T(t)} d(q,B) = \lim_{t\to\infty} \min_{E_q(k\varphi+t\mathbf{1}) = I(k\varphi+t\mathbf{1})} d(q,B) = \lim_{t\to\infty} \eta(k\varphi + t\mathbf{1})$.

Keep $\varphi \in \mathbb{R}^S$ being fixed. Let $k' \geq k > 0$ be given. Let $T$ be defined as above for $k$, and $T'$ be analogously defined for $k'$. Suppose that $p \in T(\infty)$ and $d(p,B) = \min_{T(\infty)} d(q,B)$. Similarly, suppose that $p' \in T'(\infty)$ and $d(p,B) = \min_{T'(\infty)} d(q,B)$. We would like to check that $d(p,B) \leq d(p',B)$. Note that $E_p\varphi \geq E_{p'}\varphi$. If $E_p\varphi = E_{p'}\varphi$, then $p' \in T(\infty)$, and thus $\min_{T(\infty)} d(q,B) \leq d(p',B) = \min_{T'(\infty)} d(q,B)$. If $E_p\varphi > E_{p'}\varphi$, then pick $q' \in B$ such that $d(p',B) = d(p',q')$. Since $E_{q'}\varphi \geq J(\varphi) \geq E_p\varphi$, then there uniquely exists $\alpha \in [0,1)$ such that $E_{\alpha p' + (1-\alpha)q'}\varphi = E_p\varphi$, i.e., $\alpha p' + (1-\alpha)q' \in T(\infty)$. Thus $d(p,B) \leq d(\alpha p' + (1-\alpha)q', B) \leq d(\alpha p' + (1-\alpha)q', q') = \alpha d(p',q') < d(p',q') = d(p',B)$, as desired.

Fourth, $\lim_{k\searrow 0}\lim_{t\to\infty} \eta(k\varphi + t\mathbf{1}) = 0$. Fix $\varphi \in \mathbb{R}^S$. Let $p \in B$ and $p' \in \Delta$ be given such that $E_p\varphi = J(\varphi)$ and $E_{p'}\varphi = \min_S \varphi(s)$. If $E_p\varphi = E_{p'}\varphi$, then for all $k > 0$ and $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] = E_p\varphi$. That is, for all $k > 0$ and $t \in \mathbb{R}$, $p \in D(k\varphi + t\mathbf{1})$, and thus $\min_{D(k\varphi+t\mathbf{1})} d(q,B) = 0$ since $p \in B$ as well. Hence, $\lim_{k\searrow 0}\lim_{t\to\infty} \eta(k\varphi + t\mathbf{1}) = 0$. If $E_p\varphi > E_{p'}\varphi$, then for each $k > 0$ there exists a unique $\alpha_k \in [0,1]$ such that $E_{\alpha_k p' + (1-\alpha_k)p}\varphi = \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$. Thus $\lim_{k\searrow 0} \alpha_k = \lim_{k\searrow 0} \frac{1}{E_{p'}\varphi - E'_p\varphi}[\lim_{t\to\infty} \frac{1}{k}(I(k\varphi + t\mathbf{1}) - t) - E_p\varphi] = \frac{E_p\varphi - E_p\varphi}{E_{p'}\varphi - E_p\varphi} = 0$. Combining the results above, we have that for each $k > 0$,

$$
\begin{aligned}
0 &\leq \lim_{k \searrow 0} \lim_{t\to\infty} \eta(k\varphi + t\mathbf{1}) \leq \lim_{t\to\infty} \eta(k\varphi + t\mathbf{1}) \\
&= \min_{\{q\in\Delta|\ \lim\limits_{t\to\infty} \frac{1}{k}[I(k\varphi+t\mathbf{1})-t]\}} d(q, B) \leq d(\alpha_k p' + (1-\alpha_k p), p) = \alpha_k d(p', p).
\end{aligned}
$$

By taking the limit, we the desired result.

For necessity, we only check A.3, others are easy. Suppose that $\langle u, B, \eta\rangle$ is a variant constraint representation of $\succsim$ satisfying the conditions in Theorem 1(2). Let $I : \mathbb{R}^S \to \mathbb{R}$ be defined as in (2.6). It suffices to show that $I$ is continuous. Let $\varphi \in \mathbb{R}^S$ be given. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim\limits_{n\to\infty} \varphi_n = \varphi$. If $\varphi$ is constant, then

$$
\begin{aligned}
&\lim_{n\to\infty} |I(\varphi_n) - I(\varphi)| \leq \lim_{n\to\infty} \sup_S |\varphi_n(s) - I(\varphi)| \\
&= \lim_{n\to\infty} \sup_S |\varphi_n(s) - \varphi(s)| = 0,
\end{aligned}
$$

as desired. Suppose that $\varphi$ is not constant. Define $W : R^S \rightrightarrows \Delta$ as a correspondence by $W(\phi) = \{q \in \Delta | d(q, B) \leq \eta(\phi)\}$ for all $\phi \in \mathbb{R}^S$. It suffices to show that $W$ is continuous at $\varphi$. For upper hemicontinuity, let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements in $\Delta$ such that for each $n \in \mathbb{Z}_+$, $p_n \in W(\varphi_n)$ and $\lim\limits_{n\to\infty} p_n = p$ for some $p \in \Delta$. Since $\eta$ is continuous at $\varphi$, then $\lim\limits_{n\to\infty} \eta(\varphi_n) = \eta(\varphi)$. Thus $d(p, B) = \lim\limits_{n\to\infty} d(p_n, B) \leq \lim\limits_{n\to\infty} \eta(\varphi_n) = \eta(\varphi)$. Hence, $p \in W(\varphi)$. For lower hemicontinuity, let $p \in W(\varphi)$ be given. If $d(p, B) < \eta(\varphi)$, then $p \in W(\varphi_n)$ for all sufficiently large $n$, and we are done. Suppose that $d(p, B) = \eta(\varphi)$. If $p \in B$, then $p \in W(\varphi_n)$ for all $n$, and we are done. If $p \notin B$, then $\eta(\varphi) > 0$. Suppose that $p' \in B$ and $d(p, p') = \eta(\varphi)$. Let $\{\epsilon_j\}_{j=1}^{\infty}$ be a sequence of real numbers such that for all $j \in \mathbb{Z}_+$, $\epsilon_j \in (0, \eta(\varphi))$ and $\lim\limits_{j\to\infty} \epsilon_j = 0$. For each $j \in \mathbb{Z}_+$, pick $n_j \in \mathbb{Z}_+$ such that $|\eta(\varphi_{n_j}) - \eta(\varphi)| < \epsilon_j$ and $n_{j'} > n_j$ for all $j' > j$ in $\mathbb{Z}_+$. Define $\alpha_j = 1 - \frac{\epsilon_j}{\eta(\varphi)}$ for all $j \in \mathbb{Z}_+$. Then $\alpha_j \in (0, 1)$ for all

$j \in \mathbb{Z}_+$ and $\lim_{j\to\infty} \alpha_j p + (1-\alpha_j)p' = p$. For all $j \in \mathbb{Z}_+$, $d(\alpha_j p + (1-\alpha_j)p', B) \le \alpha_j d(p, p') = \eta(\varphi) - \epsilon_j < \eta(\varphi_{n_j})$, and thus $\alpha_j p + (1-\alpha_j)p' \in W(\varphi_{n_j})$.

Lastly, for the uniqueness of the representation, let $< u', B', \eta' >$ be another variant constraint representation of $\succsim$ satisfying Theorem 1(2). Let $I' : \mathbb{R}^S$ be defined as in (2.6). Since both $u$ and $u'$ are affine functions representing preferences on constant acts. Then there exist $a > 0$ and $b \in \mathbb{R}$ such that $u'(x) = au(x) + b$ and $I'(a\varphi + b\mathbf{1}) = aI(\varphi) + b$ for all $x \in X$ and $\varphi \in \mathbb{R}^S$. Suppose for the sake of a contradiction that $B \neq B'$. Suppose further without loss of generality that $p \in B \setminus B'$. Then by a standard separation theorem, there exists $\varphi \in \mathbb{R}^S \setminus \mathbb{R}\mathbf{1}$ such that $E_p\varphi < \min_{B'} E_q\varphi$. Because of the properties of $\eta$, we have

$$\lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] = \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[\min_{\{q\in\Delta | d(q,B)\le\eta(k\varphi+t\mathbf{1})\}} E_p(k\varphi + t\mathbf{1}) - t]$$
$$= \lim_{k\searrow 0}\lim_{t\to\infty} \min_{\{q\in\Delta | d(q,B)\le\eta(k\varphi+t\mathbf{1})\}} = \min_{B} E_q\varphi < \min_{B'} E_q\varphi.$$

On the other hand,

$$\lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] = \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[\frac{I'[ak\varphi + (at+b)\mathbf{1}]}{a} - t]$$
$$= \lim_{k\searrow 0}\lim_{t\to\infty} \min_{\{q\in\Delta | d(q,B')\le\eta'(ak\varphi+(at+b)\mathbf{1})\}} = \min_{B'} E_q\varphi$$

which is a contradiction. Hence, $B = B'$.

For the uniqueness of distance bound, let $\varphi \in \mathbb{R}^S$ and $p, p' \in \Delta$ be given such that $I(\varphi) = E_p\varphi > \min_S \varphi(s) = E_{p'}\varphi$, and $d(p, B) \le \eta(\varphi)$. Suppose without loss of generality that $\eta(\varphi) < \eta'(a\varphi + b)$. Then there exists $\epsilon \in (0, 1)$ such that $d(\epsilon p + (1-\epsilon)p', B) \le \eta'(a\varphi + b)$. Thus $I'(a\varphi + b) \le E_{\epsilon p+(1-\epsilon)p'}[a\varphi + b] < aI(\varphi) + b$, which is a contradiction as desired.

**Proof of Theorem 2.** The idea of the proof is similar to that of Theorem 1. We only check the sufficiency, the necessity and uniqueness are easy. Let $u : X \to \mathbb{R}$ and $I : \mathbb{R}^S \to \mathbb{R}$ be given as in Lemma 7. We define $J : \mathbb{R}^S \to \mathbb{R}$ by

$$J(\varphi) = \lim_{k\to\infty} \lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$$

for all $\varphi \in \mathbb{R}^S$.

Note that for all $\varphi \in \mathbb{R}^S$, $k > 0$ and $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi - t\mathbf{1}) + t] \in [\min_S \varphi(s), \max_S \varphi(s)]$. For all $\varphi \in \mathbb{R}^S$ and $k > 0$, $\frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$ weakly decreases in $t$. Moreover, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$ weakly decreases in $k$. To see that, let $k' \geq k > 0$ be given. Suppose for the sake of a contradiction that $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbf{1}) + t] < \lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi - t\mathbf{1}) + t]$. Thus there exists $\bar{t}$ such that for all $t, t' \geq \bar{t}$, $\frac{1}{k}[I(k\varphi - t\mathbf{1}) + t] < \frac{1}{k'}[I(k'\varphi - t'\mathbf{1}) + t']$, that is,

$$I(k\varphi - t\mathbf{1}) < \frac{k}{k'} I(k'\varphi - t'\mathbf{1}) + \frac{k}{k'} t' - t. \tag{2.17}$$

Pick $t, t' \geq \bar{t}$ such that $\frac{k}{k'}t' + (\frac{k}{k'} - 1)I(k'\varphi - t'\mathbf{1}) = t$. Thus $k\varphi - t\mathbf{1} = \frac{k}{k'}I(k'\varphi - t'\mathbf{1}) + (1 - \frac{k}{k'})I(k'\varphi - t'\mathbf{1})\mathbf{1}$. Since $I$ is quasi-concave and normalized, then $I(k\varphi - t\mathbf{1}) \geq \frac{k}{k'}I(k'\varphi - t'\mathbf{1}) + (1 - \frac{k}{k'})I(k'\varphi - t'\mathbf{1}) = \frac{k}{k'}I(k'\varphi - t'\mathbf{1})$, which contradicts (2.17). The above properties imply that $J$ is well defined.

Similar to the proof of Theorem 1, $J$ is normalized, weakly increasing, constant additive, positive homogeneous of degree 1 and superadditive. We only check superadditivity. Suppose the contrary that there exist $\varphi, \varphi' \in \mathbb{R}^S$, $J(\varphi + \varphi') < J(\varphi) + J(\varphi')$. Thus $J(\frac{1}{2}\varphi + \frac{1}{2}\varphi') < \frac{1}{2}J(\varphi) + \frac{1}{2}J(\varphi')$. Hence there exists $k > 0$ and $\bar{t} \in \mathbb{R}$ such that for all $t, t' \in \mathbb{R}$ and $t'' \geq \bar{t}$, $\frac{1}{k}[I(k(\frac{1}{2}\varphi + \frac{1}{2}\varphi') - t''\mathbf{1}) + t''] < \frac{1}{2k}[I(k\varphi - t\mathbf{1}) + t] + \frac{1}{2k}[I(k\varphi - t'\mathbf{1}) + t']$. Rearranging the terms, we get that

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbf{1}) < \frac{1}{2}I(k\varphi - t\mathbf{1}) + \frac{1}{2}I(k\varphi' - t'\mathbf{1}) + \frac{t + t'}{2} - t''. \tag{2.18}$$

Pick $t, t' \geq \bar{t}$ such that $I(k\varphi - t\mathbf{1}) = I(k\varphi' - t'\mathbf{1})$. Define $t'' = \frac{t+t'}{2}$. Note that $t'' \geq \bar{t}$ and $\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbf{1} = \frac{1}{2}(k\varphi - t\mathbf{1}) + \frac{1}{2}(k\varphi' - t'\mathbf{1})$. Since $I$ is superadditive, then

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbf{1}) \geq \frac{1}{2}I(k\varphi - t\mathbf{1}) + \frac{1}{2}I(k\varphi' - t'\mathbf{1}),$$

which contradicts (2.18).

By Gilboa and Schmeidler (1989), there exists a unique non-empty closed convex set $C \subseteq \Delta$ such that $J(\varphi) = \min_C E_p\varphi$ for all $\varphi \in \mathbb{R}^S$. Fix $\varphi \in \mathbb{R}^S$. Then $I(\varphi) \in [\min_C E_p\varphi, \max_C E_p\varphi]$. We only check the upper bound. Let $t \in \mathbb{R}$ be given such that $I(\varphi) = I(-\varphi + t\mathbf{1})$. Since $I$ is quasi-concave, then $I(\frac{\varphi}{2} + \frac{-\varphi+t\mathbf{1}}{2}) \geq \frac{1}{2}I(\varphi) + \frac{1}{2}I(-\varphi + t\mathbf{1}) \geq \frac{1}{2}I(\varphi) + \frac{1}{2}\min_C E_p(-\varphi) + \frac{t}{2}$. Since $I$ is normalized, then $\frac{t}{2} \geq \frac{1}{2}I(\varphi) + \frac{1}{2}\min_C E_p(-\varphi) + \frac{t}{2}$. Thus, $I(\varphi) \leq -\min_C E_p(-\varphi) = \max_C E_p\varphi$.

Define $\lambda : \mathbb{R}^S \to [0, 1]$ by $\lambda(\varphi) = 1$ if $\min_C E_p\varphi = \max_C E_p\varphi$, and $\lambda(\varphi) = \frac{\max_C E_p\varphi - I(\varphi)}{\max_C E_p\varphi - \min_C E_p\varphi}$ otherwise. Note that $I(\varphi) = \lambda(\varphi)\min_C E_p\varphi + (1 - \lambda(\varphi))\max_C E_p\varphi$ for all $\varphi \in \mathbb{R}^S$.

Using the properties of $I$, it is easy to verify that $\lambda(\varphi + t\mathbf{1})$ weakly decreasing in $t$ for all $\varphi \in \mathbb{R}^S$, $\lim_{t\to\infty} \lambda(k\varphi - t\mathbf{1})$ weakly increases in $k$ when $k > 0$ for all $\varphi \in \mathbb{R}^S$, $\lim_{k\to\infty}\lim_{t\to\infty} \lambda(k\varphi - t\mathbf{1}) = 1$, and $\lambda$ is continuous on $\{\varphi \in \mathbb{R}^S | \min_C E_p\varphi < \max_C E_p\varphi\}$.

**Proof of Proposition 1.** The proof mainly makes use of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011)'s representation theorems for uncertainty averse preferences. We first quote their result.

**A.2′. Risk Independence.** For all $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$x \sim y \Rightarrow \alpha x + (1 - \alpha)z \gtrsim \alpha y + (1 - \alpha)z.$$

**Theorem 7.** [8] *The following two statements are equivalent.*

*(1) A preference* $\succsim$ *on* $\mathcal{F}$ *satisfies A.1, A.2′, A.3 - A.6.*

*(2) There exists an affine onto function* $u : X \to \mathbb{R}$*, and a lower semicontinuous function* $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ *such that*

*(i)* $G$ *is quasi-convex on* $\mathbb{R} \times \Delta$*;*

*(ii)* $G(\cdot, p)$ *is increasing for all* $p \in \Delta$*;*

*(iii)* $\inf_{\Delta} G(t, p) = t$ *for all* $t \in \mathbb{R}$*;*

*(iv) for* $I : \mathbb{R}^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{\Delta} G(E_p\varphi, p) \ for \ all \ \varphi \in \mathbb{R}^S,$$

$I$ *is continuous on* $\mathbb{R}^S$*; (v) for* $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)) \ for \ all \ u \in \mathcal{F},$$

$V$ *represents* $\succsim$.

*Moreover,* $u$ *is unique up to a positive affine transformation. For each* $u$*,* $G$ *is uniquely given by*

$$G(t, p) = \sup_{\mathcal{F}}\{u(x_f) : E_p u(f) \leq t\} \ for \ all \ (t, p) \in \mathbb{R} \times \Delta. \tag{2.19}$$

For the sufficiency, suppose that $\succsim$ satisfies A.1 - A.6. Thus $\succsim$ satisfies A.2′ (see e.g. Lemma 28 of Maccheroni, Marinacci and Rustichini (2006)). Hence, there exists $u : X \to \mathbb{R}$ and $G : \mathbb{R}\times\Delta \to (-\infty, \infty]$ which satisfy the conditions in Theorem 7. Moreover, $u$ is unique up to a positive affine transformation and $G$ is uniquely given by (2.19).

[8] This theorem combines Theorem 3 and Theorem 5 in Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

Define $c : \mathbb{R} \times \Delta \to [0, \infty]$ by

$$c(t, p) = G(t, p) - t \; for \; all \; (t, p) \in \mathbb{R} \times \Delta.$$

We check that $< u, c >$ is a DAUA variational representation of $\succsim$. By definition and the corresponding properties of $G$, $c$ is lower semicontinuous on $\mathbb{R} \times \Delta$, $c(t, p) + t$ is quasi-convex on $\mathbb{R} \times \Delta$, and $\inf_{\Delta} c(t, p) = 0$ for each $t \in \mathbb{R}$. Moreover, for $I$ defined as in (2.9), $I$ is continuous on $\mathbb{R}^S$, and $V : \mathcal{F} \to \mathbb{R}$ represents $\succsim$.

To see that $c(t, p)$ is weakly increasing in $t$, let $p \in \Delta$ and $t' \geq t$ in $\mathbb{R}$ be given. For any $f \in \mathcal{F}$ such that $E_p u(f) \leq t$, there exists $f' \in \mathcal{F}$ such that $u(f') = u(f) + (t' - t)\mathbf{1}$ and thus $E_p u(f') \leq t'$. Let $I$ be given as in (2.9). Note that it is normalized and constant superadditive. Thus $u(x_{f'}) - t' = I(u(f')) - t' = I(u(f) + t' - t) - t' \geq I(u(f)) + t' - t - t' = I(u(f)) - t = u(x_f) - t$. Since $G$ is given by (2.19), then by definition $c(t', p) \geq c(t, p)$.

The necessity and uniqueness are routine. We only check the necessity for A.2. It suffices to show that $I$ is constant superadditive. Fix $\varphi \in \mathbb{R}^S$, $p \in \Delta$ and $t > 0$. Then $I(\varphi + t\mathbf{1}) = \min_{\Delta}[E_p(\varphi + t\mathbf{1}) + c(E_p(\varphi + t\mathbf{1}), p)] = \min_{\Delta}[E_p\varphi + c(E_p\varphi + t, p)] + t \geq \min_{\Delta}[E_p\varphi, c(E_p\varphi, p)] + t = I(\varphi) + t$, as desired.

# Chapter 3

# Aspiration and confidence under uncertainty

## 3.1 Introduction

The fundamental work of Savage (1954) establishes a beautiful theory to derive the subjective probability distribution from a decision maker's preferences.[1] For example, if the decision maker prefers to bet on event $A$ than event $B$, then $A$ is revealed to be more likely for him than $B$. This approach is challenged by Ellsberg (1961)'s famous thought experiments. One example of Ellsberg (1961) is to consider an urn with 90 balls in it, 30 red and 60 either black or green. A ball is drawn from this urn. You are asked first to choose from betting on red and betting on green. Next, you are asked to choose from betting on "either red or green" and "either black or green". For each draw, the payoff structure is the same: you get 100 dollars if your bet is right and nothing if it is wrong. As Ellsberg (1961) suggests and many later experiments confirm, most decision makers prefer to bet on red in the first case and "either black or green" in the second. If a decision maker does have a subjective belief, then his first choice reveals that the probability of the ball being green is less than $\frac{1}{3}$, while the second choice reveals it more than $\frac{1}{3}$.

This example shows that subjective probability may not exist. In this decision problem, there exists ambiguity, and ambiguity matters for a decision maker's choices. Indeed, the proportion of green balls is *unknown* and the decision maker has a aversion to betting on the ambiguous events. The literature goes further by asking mainly two questions: (1)

[1] The early idea goes back to Ramsey (1931) and de Finetti (1937).

Can we distinguish between ambiguous and unambiguous events, and derive a decision maker's subjective probability on unambiguous events from his preference? (2) How does a decision maker make choices under ambiguity?

In responding to the first question, most studies define an ambiguous event via the "inconsistency" or reversal of preferences in some fashion, and define unambiguous events and acts accordingly.[2] Axioms are imposed on preferences to deliver a subjective probability over the unambiguous events and also probabilistic sophistication behavior on unambiguous acts. Many criticisms arise when researchers find examples[3] where some obviously ambiguous events are identified as unambiguous and some apparently unambiguous events are identified as ambiguous according to the definitions. One important reason for such misidentification proposed by Klibanoff, Marinacci and Mukerji (2011) is that some definition confounds ambiguity with ambiguity attitude. The inconsistency of preference may result from a decision maker's changing ambiguity attitude rather than the ambiguity of some event. Nehring (2006) finds that even the consistency of preference may have nothing to do with the existence of a subjective probability on relevant events.

In responding to the second question, many models have proposed different axioms on preferences to characterize different decision rules.[4] Implicitly or explicitly, most decision rules base the interpretation on two factors: the ambiguity that a decision maker feels, and his attitude toward this ambiguity. For example, Gilboa and Schmeidler (1989)'s "maxmin expected utility" (MEU) models the ambiguity as a set of priors and presents a decision maker's ambiguity attitude by his taking the minimum expected utilities over these priors. While ambiguity reflects a decision maker's personal perception of the underlying situation,

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[2]E.g., see Nehring (1999), Epstein and Zhang (2001), Ghirardato and Marinacci (2001), Zhang (2002), Klibanoff, Marinacci and Mukerji (2005).

[3]E.g., see Klibanoff, Marinacci and Mukerji (2011), Nehring (2006).

[4]E.g., see Gilboa and Schmeidler (1989), Schmeidler (1989), Klibanoff, Marinacci and Mukerji (2005), Marinacci, Maccheroni and Rustichini (2006), Strzalecki (2011).

ambiguity attitude shows how he responds to it. His preference is determined by both factors, but they are subjective and unobservable for the outside modeler. Hence, with only the choice data, there is some arbitrariness in explaining a decision maker's behavior by these two free variables. For instance, in contrast to the MEU model, the decision maker may in reality have a larger set of priors and evaluate an act by a weighted average of the minimum and maximum expected utility over this set of priors, rather than always considering the worst scenario. Although Ghirardato, Maccheroni and Marinacci (2004) characterize the $\alpha$-MEU representation[5] and claim that their model completely separates ambiguity and ambiguity attitude, Eichberger, Grant, Kelsey and Koshevoy (2011) shows that their identification of the set of priors summarizing a decision maker's perceived ambiguity fails for general $\alpha$-MEU preferences with $\alpha \neq 0$ or 1.

The difficulty of answering both questions is same. Can we identify a decision maker's perceived ambiguity and his ambiguity attitude with only the data of choices? Nehring (2006) conjectures that the purely behavioral approach itself may have deep-seated limitation in identifying subjective beliefs under ambiguity. Klibanoff, Marinacci and Mukerji (2005) represent a decision maker's perception about ambiguity as a probability over priors, and resort to the “second-order" acts, i.e., the bets on the priors, to reveal his belief over priors. They notice that there is a question whether the preference with respect to the second-order acts are observable: “When it is not evident we may need something richer than behavioral data, perhaps cognitive data or thought experiments, to help us reveal the decision maker's belief over Δ [the priors].”

This paper achieves the goal of identifying a decision maker's perceived ambiguity by

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[5]The functional form is

$$V(f) = \alpha \min_C E_p u(f) + (1 - \alpha) \max_C E_p u(f)$$

where $f$ is an act, $u(f)$ is the state-contingent utility profile, $E_p u(f)$ is the expectation of $u(f)$ with respect to a prior $p$, $C$ is a non-empty closed convex set of priors, and $\alpha \in [0, 1]$ is a weight parameter.

explicitly introducing a *confidence order* in addition to the preference order. The confidence order can be obtained from psychological data. It ranks the degree of a decision maker's confidence in aspiring a particular return from an act. More precisely, an act generates different expected payoffs under different priors. Given an act and an aspired expected payoff, if the expected payoff of the act under a prior is no less than the aspiration level, then the prior is called a *supportive* prior. The higher the aspiration level, the less the supportive priors, and the lower the confidence in achieving the aspiration. The confidence depends on the decision maker's belief about the likelihood of the set of supportive priors. It is similar to the second-order belief over priors proposed by Klibanoff, Marinacci and Mukerji (2005). However, under the axioms imposed on this confidence order, a capacity instead of a probability over the priors is elicited. Capacity is a generalization of probability and is accepted in the literature as a more relevant notion to model the belief under ambiguity.[6]

Together with the confidence order, this paper axiomatizes a class of preferences that base the evaluation of acts on two criteria: a decision maker's aspiration from the act, and his confidence in achieving this aspiration level. As discussed above, the confidence in aspiration decreases in the aspiration level. Each act corresponds to a trade-off between aspiration and confidence. The preferences that we characterize evaluate an act by the optimal combination of aspiration and confidence according to another aggregating preference over the two-criteria plane. The aggregating preference is endogenously determined from the preference over acts.

Although several papers compare the degrees of decision makers' ambiguity aversion based only on their preferences, such comparison is meaningful only when the decision makers perceive the same ambiguity; otherwise, ambiguity and ambiguity attitude will be

[6]See eg. Schmeidler (1989).

confounded. In this paper, since decision makers' perception about ambiguity is revealed by confidence orders, the comparison of ambiguity attitude can be obtained. A decision maker is more ambiguity averse than another if (1) he perceives the same ambiguity as the other, that is, they have the identical confidence order, and (2) at each aspiration level, according to the aggregating preference, he requires more degree of confidence to achieve the same level of satisfaction, that is, the indifference curves of his aggregating preference lie above those of the other.

An important feature of this class of preferences is that it allows a decision maker's ambiguity attitude to change across acts. Most current literature assumes to some extent that a decision maker always display the same degree of ambiguity aversion.

## 3.2 Setup

We denote by $\mathbb{R}$ the set of all reals, $\mathbb{R}_+$ the set of all positive reals and $\mathbb{Z}_+$ the set of positive integers. Let $S$ be a finite set of *states of the world*. Suppose that $|S| \geq 2$ where $|\cdot|$ denotes the cardinality of a set. A subset of $S$ is called an *event*. Let $\Delta(S)$ be the set of all probability measures on $S$. We identify $\Delta(S)$ with the standard $|S|-1$ dimensional simplex in $\mathbb{R}^{|S|}$, i.e., the set $\{(p_1, ..., p_{|S|}) \in \mathbb{R}^{|S|} \mid \sum_{i=1}^{|S|} p_i = 1 \text{ and } p_i \geq 0 \text{ for all } i\}$.

Let $X$ be a set of *outcomes*. Suppose that $X$ is a connected metric space. Let $\Delta(X)$ be the set of all Borel probability measures on $X$. An element $l \in \Delta(X)$ is called a *lottery* on $X$. For all $l \in \Delta(X)$ and $x \in X$, we write $l(x)$ to denote $l(\{x\})$. A lottery $l \in \Delta(X)$ is a *simple lottery* if $l(x) \neq 0$ for finitely many $x \in X$, and $\sum_{x \in X} l(x) = 1$. Let $\mathcal{L}_0$ be the set of all simple lotteries. For all $l \in \mathcal{L}_0$, denote by supp$(l)$ the support of $l$, i.e. the set $\{x \in X \mid l(x) > 0\}$.

Let $f : S \to X$ be a (Savage) *act* which specifies an outcome in each state. Let $\mathcal{F}_0 = X^S$ be the set of all the (Savage) acts. Consider the set $\mathcal{F}_1 = \mathcal{F}_0 \times \Delta(S)$. A pair $(f, p) \in \mathcal{F}_1$ is an *informed act* which denotes an act $f$ with the given information that the probability over $S$ is $p$. For example, consider an Ellsberg's urn containing black and white balls with *known* proportion. Betting on the color of a ball drawn from this urn is an act with given probabilistic information. Alternatively, we may also think of it as an act with postulated prior in the mind of decision maker, and he may compare the acts assoicated with different prior information as a thought experiment.

Given $(f, p) \in \mathcal{F}_1$, $l \in \Delta(X)$ is said to be *generated* by $(f, p)$ if $l(x) = \sum_{f(s)=x} p_s$ for all $x \in X$. Let $\mathcal{L}_1$ be the set of all the lotteries generated by some element in $\mathcal{F}_1$. Endow $\mathcal{L}_1$ with the weak topology induced by the collection of real-valued functions on $\mathcal{L}_1$ of the form $\int \eta dl$ for all $l \in \mathcal{L}_1$, where $\eta$ is a continuous real-valued function on $X$. Note that a sequence $\{l_n\}_{n=1}^{\infty}$ of elements in $\mathcal{L}_1$ converges to $l \in \mathcal{L}_1$ with respect to this topology if and only if $\lim_{n\to\infty} \int \eta dl_n = \int \eta dl$ for every continous real-valued function $\eta$ on $X$.

Given $l_1, l_2 \in \mathcal{L}_0$ and $\lambda \in [0, 1]$, we define their convex combination $\lambda l_1 + (1 - \lambda)l_2$ as a lottery in $\mathcal{L}_0$ such that $[\lambda l_1 + (1 - \lambda)l_2](Y) = \lambda l_1(Y) + (1 - \lambda)l_2(Y)$ for all Borel sets $Y \subseteq X$. For simplicity we write $l_1\lambda l_2$ instead of $\lambda l_1 + (1-\lambda)l_2$. Note that $\mathcal{L}_0$ is closed under convex combination, but $\mathcal{L}_1$ is not. Given $l_1, l_2 \in \mathcal{L}_0$ and $\lambda \in (0, 1)$, $l_1\lambda l_2 \in \mathcal{L}_1$ if and only if $|\text{supp}(l_1\lambda l_2)| \leq |S|$. More precisely, $\mathcal{L}_1$ is the set of all simple lotteries which assign positive probability to at most $|S|$ elements in $X$.

We model a decision maker's preference $\succsim$ as a binary relation on $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$. Let $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$ as usual. Given $x \in X$, let $f_x$ denote an act in $\mathcal{F}_0$ such that $f_x(s) = x$ for all $s \in S$, and let $l_x$ denote a lottery in $\mathcal{L}_1$ such that $l_x(x) = 1$. We call $f_x$ a *constant act* and $l_x$ a *degenerate lottery* where $x \in X$. Given $x, y \in X$ and $A \subseteq S$, let $xAy$ denote an act in $\mathcal{F}_0$ such that $xAy(s) = x$ for all $s \in A$ and $xAy(s) = y$ for all $s \in S \setminus A$.

Lastly, given a set $Z$ and an order $\trianglerighteq$ on $Z$, let $\max Z = \{z \in Z \mid z \trianglerighteq z' \text{ for all } z' \in Z\}$. We write $\max Z \trianglerighteq z$ if $z' \trianglerighteq z$ for all $z' \in \max Z$; $\min Z$ and $z \trianglerighteq \min Z$ are similarly defined. A real-valued function $T$ on $Z$ is said to represent $\trianglerighteq$ on $Z$ if and only if $T(z) \geq T(z') \Leftrightarrow z \trianglerighteq z'$ for all $z, z' \in Z$.

This setup compromises the setting of Savage (1954) and that of Anscome and Aumann (1963). Anscome and Aumann (1963) differs from Savage (1954) in that they further assume the set of outcomes is the set of all simple lotteries over a set of prizes. The introduction of *objective* lotteries enlarges the alternatives of choice and significantly simplies the analysis. However, it is under the criticism that in many real-life situations, there is no random device to generate objective lotteries. Our model retains the advantage of Anscome and Aumann (1963)'s setting, while we provide a way to generate lotteries endogenously. We introduce the set of informed acts in addition to Savage acts. By imposing a neutrality axiom (see Section 2), the informed acts in $\mathcal{F}_1$ are essentially regarded as lotteries by the decision maker. Moreover, while Anscome and Aumann (1963) assume the existence of all simple lotteries, we only need a subset — the set of generated lotteries.

## 3.3 Expected utility representation on informed acts

We consider the following axioms on $\succsim$. In this section, we focus on the representation for preference over informed acts.

**A.1. Weak Order.** $\succsim$ is complete and transitive.

**A.2.1. Neutrality.** If $(f, p), (g, q) \in \mathcal{F}_1$ generate the same lottery, then $(f, p) \sim (g, q)$.

Each lottery in $\mathcal{L}_1$ corresponds to a set of informed acts which generate it, and A.2.1 says that these acts are equivalent with respect to $\succsim$. In the following, we would not distinguish between $\mathcal{F}_1$ and $\mathcal{L}_1$, and will interchangeably use them for convenience. We would like to obtain the expected utility representation for preference over informed acts. However, as mentioned above, the set of generated lotteries is the set of simple lotteries which assign positive probability to at most $|S|$ outcomes. It is not a convex set, so the mixture space theorem (Herstein and Milnor (1953)) does not apply. Nevertheless, the expected utility representation can still be obtained with the standard axioms (see Lemma 11 in Appendix).

**A.3.1. Continuity.** For any $l_1 \in \mathcal{L}_1$, the sets $\{l \in \mathcal{L}_1 \mid l \succsim l_1\}$ and $\{l \in \mathcal{L}_1 \mid l_1 \succsim l\}$ are closed in $\mathcal{L}_1$.

**A.4.1. Independence.** For any $l_1, l_2, l_3 \in \mathcal{L}_1$ and $\lambda \in (0,1)$, $l_1 \succ l_2$ implies that $l_1 \lambda l_3 \succ l_2 \lambda l_3$ if both $l_1 \lambda l_3$ and $l_2 \lambda l_3$ exist in $\mathcal{L}_1$.

**A.5. Unboundedness.** There exist $l_x, l_y$ in $\mathcal{L}_1$ such that (1) $l_x \succ l_y$, and (2) for all $\lambda \in (0,1)$ there are $z_1, z_2 \in X$ satisfying $l_y \succ l_{z_1} \lambda l_x$ and $l_{z_2} \lambda l_y \succ l_x$.

Axiom A.3 strengthens the usual continuity axiom in a similar way as Grandmont (1972), so that the obtained Bernoulli utility function is continuous. Axiom A.4 further enforces the Bernoulli utility function to be unbounded. This axiom is commonly used in the recent literature (see e.g. Kopylov (2001), Maccheroni, Marinacci and Rustichini (2006), Strzalecki (2011b) and Grant and Polak (2011)).[7]

Let $u : X \to \mathbb{R}$ be a utility function of outcomes. Given $f \in \mathcal{F}$, let $u(f)$ a real-valued function on $S$ assigning $u(f(s))$ to each $s \in S$. Thus, $u(f)$ transfers each act $f$ to a state-contingent utility function. Given $f \in \mathcal{F}_0$, $u : X \to \mathbb{R}$ and $p \in \Delta(S)$, let $E_p u(f)$ denote the expected value of $u(f)$ with respect to $p$, i.e., $E_p u(f) = \sum_{s \in S} u(f(s)) p_s$. Given $l \in \mathcal{L}_0$ and $u : X \to \mathbb{R}$, let $E_l u$ denote the expected value of $u$ with respect to $l$, i.e., $E_l u = \sum_{x \in X} u(x) l(x)$. Note that if $(f,p), (g,q) \in \mathcal{F}_1$ generate the same lottery $l \in \mathcal{L}_1$, then $E_l u = E_p u(f) = E_q u(g)$.

[7]They impose unboundedness on the preference over constant acts. Our unboundedness axiom is the same as theirs since later we will also impose a neutrality axiom to “equalize" degenerate lotteries and constant acts (see Axiom A.2.2).

**Lemma 8.** *Suppose that $\succsim$ satisfies A.1. Then $\succsim$ satisfies A.2.1, A.3.1 and A.4.1 if and only if there is a continuous function $u : X \to \mathbb{R}$ such that for all $(f, p), (g, q) \in \mathcal{F}_1$, $(f, p) \succsim (g, q) \Leftrightarrow E_p u(f) \geq E_q u(g)$. Moreover, $u$ is unique up to a positive affine transformation. The set $u(X)$ is $\mathbb{R}$ if and only if $\succsim$ satisfies A.5 in addition.*

## 3.4 Confidence order

Consider another binary relation $\succsim'$ on $\mathcal{M} = (\mathcal{L}_1 \times X) \cup (\mathcal{F}_0 \times \mathcal{L}_1)$, with $\succ'$ and $\sim'$ denoting its asymmetric and symmetric parts respectively. A pair $(l, x) \in \mathcal{L}_1 \times X$ stands for the confidence that a decision maker has in aspiring a lottery $l$ to achieve an outcome at least as good as $x$. A pair $(f, l) \in \mathcal{F}_0 \times \mathcal{L}_1$ stands for the confidence that he has in aspiring an act $f$ to be at least as good as $l$. The confidence order compares the confidence levels that a decision maker has to aspire $l$ from $f$ or $x$ from $l$ for all $(f, l) \in \mathcal{F}_0 \times \mathcal{L}_1$, $(l, x) \in \mathcal{L}_1 \times X$. To reflect the nature of a confidence order, consider the axioms below.

Given $l \in \mathcal{L}_1$, let $L_l : X \to [0, 1]$ denote the decumulative distribution function such that $L_l(x) = \sum_{l_y \succsim l_x} l(y)$ for all $x \in X$.

**A.1′. Order.** $\succsim'$ is complete and transitive.

**A.2′. Continuity.** For any $x \in X$, $f \in \mathcal{F}_0$ and $l_1, l_2, l_3 \in \mathcal{L}_1$ such that $l_1 \lambda l_3 \in \mathcal{L}_1$ for some $\lambda \in (0, 1)$,

(1) if $(l_1, x) \succ' (f, l_2) \succ' (l_3, x)$, then there exist $\alpha, \beta \in (0, 1)$ such that $(l_1 \alpha l_3, x) \succ' (f, l_2) \succ' (l_1 \beta l_3, x)$;

(2) if $(f, l_1) \succ' (l_2, x) \succ' (f, l_3)$, then there exists $\beta \in (0, 1)$ such that $(l_2, x) \succ' (f, l_1 \beta l_3)$, and moreover, when $(f, l_1) \notin \max \mathcal{M}$ or $(f, l_3) \notin \min \mathcal{M}$, there exists $\alpha \in (0, 1)$ such that $(f, l_1 \alpha l_3) \succ' (l_2, x)$.

**A.3.1′. Monotonicity.** Let $x_1, x_2 \in X$, $f_1, f_2 \in \mathcal{F}_0$ and $l_1, l_2 \in \mathcal{L}_1$ be given. Then

(1) $(l_1, x_1) \succsim '(l_2, x_2)$ if and only if $L_{l_1}(x_1) \geq L_{l_2}(x_2)$;

(2) if $(f_2, p) \succsim l_2$ implies $(f_1, p) \succsim l_1$ for each $p \in \Delta(S)$, then $(f_1, l_1) \succsim '(f_2, l_2)$.

**A.4′. Neutrality.** For any $x_1, x_2 \in X$, $(f_{x_1}, l_{x_2}) \sim' (l_{x_1}, x_2)$.

Axiom A.1′ is standard. We first look at A.3.1′. It says that for any objective lottery $l$, the confidence of aspiring $x$ from $l$ depends on the likelihood to obtain $x$ or better outcomes. Note that here we implicitly assume that the preference over outcomes is induced by the preference over degenerate lotteries. On the other hand, for each $(f, l) \in \mathcal{F}_0 \times \mathcal{L}_1$, if $p \in \Delta(S)$ and $(f, p) \succsim l$, then we say that $p$ is a emphsupporting prior for $(f, l)$. Axiom A.3.1′ says that for any $(f_1, l_1), (f_2, l_2) \in \mathcal{F}_0 \times \mathcal{L}_1$, if every prior supporting $f_2$ to achieve $l_2$ also supports $f_1$ to achieve $l_1$, then the decision maker feels more confident to aspire $l_1$ from $f_1$ than to aspire $l_2$ from $f_2$.

Next, to understand A.4′, let $x_1, x_2 \in X$ be given. Note that with whatever prior, $f_{x_1}$ generates the degenerate lotter $l_{x_1}$. If $l_{x_1} \succsim l_{x_2}$, then the decision maker knows for sure that $f_{x_1}$ can achieve something at least as good as $l_{x_2}$, and thus he will have full confidence to aspire $l_{x_2}$ from $f_{x_1}$. If $l_{x_2} \succ l_{x_1}$, then he knows for sure that $f_{x_1}$ can never achieve the level of $l_{x_2}$, so he will no confidence. Similarly, he has full confidence to aspire $x_2$ from $l_{x_1}$ when $l_{x_1} \succsim l_{x_2}$, and has no confidence otherwise. Since the confidence for both $(f_{x_1}, l_{x_2})$ and $(l_{x_1}, x_2)$ depends on the ranking of $l_{x_1}$ and $l_{x_2}$, then it is natrual to assume that $(f_{x_1}, l_{x_2}) \sim'$ $(l_{x_1}, x_2)$.

Finally, A.2′ is a standard continuity axiom. In fact, it is even weaker since it does not require the right continuity in some cases. For example, if $f$ is a constant act, then as discussed above, for any $l \in \mathcal{L}_1$, the decision maker has either full confidence or no confidence to aspire $l$ from $f$. Suppose that $f = f_x$ for some $x \in X$ and $l_3 \succ l_x \sim l_1$, then he has full confidence for the aspiration level $l_1$ and have no confidence for $l_3$. In this case, for any $\alpha \in (0, 1)$, $l_1 \alpha l_3 \succ l_x$, and thus $(f, l_1 \alpha l_3) \sim' (f, l_3)$ since the decision maker has no confidence for both $l_1 \alpha l_3$ and $l_3$.

For each $(f, l) \in \mathcal{F}_0 \times \mathcal{L}_1$, let $D(f, l)$ be the set of all supporting priors for $(f, l)$. Let $\mathbb{D} = \{D(f, l) \subseteq \Delta(S) \mid (f, l) \in \mathcal{F}_0 \times \mathcal{L}_1\}$. Note that both $\emptyset$ and $\Delta(S)$ belong to $\mathbb{D}$. A *capacity*[8] on $\mathbb{D}$ is a function $c : \mathbb{D} \to \mathbb{R}$ such that (1) $c(D_1) \geq c(D_2)$ if $D_1 \supseteq D_2$, and (2) $c(\emptyset) = 0$, $c(\Delta(S)) = 1$. A capacity $c$ on $\mathbb{D}$ is *upper continuous* if $\lim_{n\to\infty} c(D_n) = c(\cap_{n=1}^{\infty} D_n)$ for any non-increasing sequence $\{D_n\}_{n=1}^{\infty}$ of elements in $\mathbb{D}$ and $\cap_{n=1}^{\infty} D_n$ in $\mathbb{D}$. It is *lower continuous* if $\lim_{n\to\infty} c(D_n) = c(\cup_{n=1}^{\infty} D_n)$ for any non-decreasing sequence $\{D_n\}_{n=1}^{\infty}$ of elements in $\mathbb{D}$ and $\cup_{n=1}^{\infty} D_n$ in $\mathbb{D}$. A capacity $c$ on $\mathbb{D}$ is *continuous* if it is both upper and lower continuous.

**Lemma 9.** *Suppose that $\succsim$ satisfies Axiom 1, 2.1 - 4.1, 5, and u is given as in Lemma 8. The following statements are equivalent.*

*(1) $\succsim'$ satisfies Axiom 1′, 2′, 3.1′, 4′.*

*(2) There exists a unique continuous capacity c on $\mathbb{D}$ such that*

$$V(\cdot, \cdot) = \begin{cases} L(x) & (l, x) \in \mathcal{L}_1 \times X \\ c(D(f, l)) & (f, l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

*represents $\succsim'$. Moreover, for each $f \in \mathcal{F}_0$, the robustness index $v_f : \mathbb{R} \to [0, 1]$ defined by $v_f(t) = V(f, l)$ if $E_p u = t$ is continuous when $v_f(\mathbb{R}) \neq \{0, 1\}$.*

Given $f \in \mathcal{F}_0$, if there exists $l \in \mathcal{L}_1$ such that $(f, l) \notin \max \mathcal{M} \cup \min \mathcal{M}$, then we say that $f$ is an *ambiguous act*, and $l$ is *ambiguous for $f$*. Otherwise, we call $f$ an *unambiguous act*. Denote by $\mathcal{F}_a$ the set of all ambiguous acts.

[8] The capacity is not defined on an algebra of $\Delta(S)$ as usual, since first the sets not in $\mathbb{D}$ is irrelavant for our purpose, and second the capacity can alwasys be extended to a whole algebra, but some of its properites like continuity may lose.

Given $A \subseteq S$, if for all $x, y \in X$, $xAy$ is an unambiguous act, then $A$ is called an *unambiguous event*. Let $\mathcal{U}$ be the set of all unambiguous events. Note that $\emptyset, S \in \mathcal{U}$.

Given a nonempty closed subset $C$ of $\Delta(S)$, we say that a capacity $c$ on $\mathbb{D}$ is *compatible with the information set* $C$ if (1) $c(D_1) \geq c(D_2)$ when $D_1,\ D_2 \in \mathbb{D}$ and $D_1 \cap C \supseteq D_2 \cap C$; and (2) for any $f \in \mathcal{F}_0$, $c(D(f,l)) > 0$ when $\max\{(f,p) \mid p \in C\} > l$, and $c(D(f,l)) < 1$ when $l > \min\{(f,p) \mid p \in C\}$. The underlying information structure captured by $C$ is that the decision maker believes that the true probability lies in $C$, but he cannot exclude any prior in $C$. In particular, if $C = \{p\}$ for some $p \in \Delta(S)$ and $D \in \mathbb{D}$, then $c(D) = 1$ when $p \in D$ and $c(D) = 0$ otherwise. In this case, the decision maker has a subjective probability $p$. Moreover, every act and every event is unambiguous.

In the following, we would like to elicit more information of the decision maker's subjective information structure from his confidence order.

Given $f_1, f_2 \in \mathcal{F}_0$ and $\lambda \in [0,1]$, define their *convex combination* $f_1\lambda f_2$ to be an act in $\mathcal{F}_0$ such that $l_{f_1\lambda f_2(s)} \sim l_{f_1(s)}\lambda l_{f_2(s)}$ for all $s \in S$. Rigorously speaking, $f_1\lambda f_2$ denotes a family of such acts, but we do not need to distinguish among them for our purpose. Given $(f_1, l_1), (f_2, l_2) \in \mathcal{F}_0 \times \mathcal{L}_1$, if $l_{f_1\frac{1}{2}f_2(s)} \sim l_1\frac{1}{2}l_2$ for all $s \in S$, we say that $(f_2, l_2)$ is a *complement pair* of $(f_1, l_1)$, and denote it by $(-f_1, -l_1)$. Again a complement pair indeed denotes a family of such pairs but they are treated as the same. Note that since $X$ is connected and unboundedness is assumed, then there always exist a convex combination of $f_1$ and $f_2$ in $\mathcal{F}_0$, and a complement pair of $(f, l)$ in $\mathcal{F}_0 \times \mathcal{L}_1$. Given $f_1, f_2 \in \mathcal{F}_0$ and $u : X \to \mathbb{R}$ as in Lemma 8, it is easy to see that $u(f_1\lambda f_2(s)) = \lambda u(f_1(s)) + (1-\lambda)u(f_2(s))$ and $u(f_1(s)) + u(-f_1(s)) = E_{l_1}u + E_{-l_1}u$ for all $s \in S$.

**A.3.2′ Strong Monotonicity.** Let $x_1, x_2 \in X$, $f_1, f_2 \in \mathcal{F}_0$ and $l_1, l_2 \in \mathcal{L}_1$ be given. Then

(1) $(l_1, x_1) \succsim' (l_2, x_2)$ if and only if $L_{l_1}(x_1) \geq L_{l_2}(x_2)$;

(2) if $(f_2, p) \succsim l_2$ implies $(f_1, p) \succsim l_1$ either for each $p \in \Delta(S)$ or for each $p \in$

$D(f_3, l_3)$ where $(f_3, l_3) \in \max \mathcal{M}$, then $(f_1, l_1) \succsim'(f_2, l_2)$.

(3) if $f \in \mathcal{F}_a$ and $l_1, l_2 \in \mathcal{L}_1$ are ambiguous for $f$, then $l_2 > l_1$ implies $(f, l_1) \succ' (f, l_2)$.

**A.5′. Belief Consistency.** If $(f, l) \in \max \mathcal{M}$ and $l' > -l$, then $(-f, l') \in \min \mathcal{M}$. If $(f, l) \in \min \mathcal{M}$ and $-l > l'$, then $(-f, l') \in \max \mathcal{M}$. If $(f_1, l_1), (f_2, l_2) \in \min \mathcal{M}$ and $\lambda \in (0, 1)$, then $(f_1 \lambda f_2, l_1 \lambda l_2) \in \min \mathcal{M}$.

Axiom A.3.2′ strengthens A.3.1′ in two aspects. First, if $(f_3, l_3) \in \max \mathcal{M}$, then the decision maker have full confidence to aspire $f_3$ to achieve $l_3$, and it is as if that he believes that the set of possible priors is contained in $D(f_3, l_3)$. Axiom A.3.2′(2) simply assumes that in this case, the decision maker does not worry about the priors outside $D(f_3, l_3)$. Thus, if for every prior $p \in D(f_3, l_3)$ supporting $f_2$ to achieve $l_2$ also supports $f_1$ to achieve $l_1$, then $(f_1, l_1) \succsim'(f_2, l_2)$. Second, if $l_1, l_2 \in \mathcal{L}_1$ are ambiguous for $f \in \mathcal{F}_a$ so that the decision maker is not sure whether or not $f$ can achieve them, then the a strictly higher aspiration level corresponds to a strictly lower confidence level.

Axiom A.5′ is a belief consistency axiom. If $(f, l) \in \max \mathcal{M}$, then it is as if that the decision maker believes that the true prior is in $D(f, l)$. Since $l' > -l$, then $D(-f, l) \subseteq \Delta(S) \setminus D(f, l)$. Thus it is natrual to assume that he has no confidence to aspire $l'$ from $-f$. Similarly, if $(f, l) \in \min \mathcal{M}$ and $-l > l'$, then $D(-f, l') \subseteq \Delta(S) \setminus D(f, l)$ and it is as if that $D(-f, l')$ includes all priors which are possibly ture. Thus, it is reasonable to assume that the decision maker has full confidence to aspire $l'$ from $-f$. Lastly, if $(f_1, l_1), (f_2, l_2) \in \min \mathcal{M}$, then it is as if that neither $D(f_1, l_1)$ nor $D(f_2, l_2)$ contains any true prior. For any $\lambda \in (0, 1)$, since $D(f_1 \lambda f_2, l_1 \lambda l_2) \subseteq D(f_1, l_1) \cup D(f_2, l_2)$, then $(f_1 \lambda f_2, l_1 \lambda l_2) \in \min \mathcal{M}$ makes sense.

Let $N \in \mathbb{Z}_+$ be give. For any $a, a' \in \mathbb{R}^N$, $a \geq a'$ ($a > a'$) if $a_n \geq a'_n$ ($a_n > a'_n$) for all $n = 1, ..., N$. We say that $T : \mathbb{R}^N \to \mathbb{R}$ is *increasing (decreasing)* if $T(a) \geq T(a')$

$(T(a) \leq T(a'))$ for all $a \geq a'$ in $\mathbb{R}^N$, and that $T$ is *strictly increasing (strictly decreasing)* if $T$ is increasing (decreasing) and $T(a) > T(a')$ $(T(a) < T(a'))$ for all $a > a'$ in $\mathbb{R}^N$.

**Lemma 10.** *Suppose that $\gtrsim$ satisfies Axiom 1, 2.1 - 4.1, 5. The following statements are equivalent.*

*(1) $\gtrsim'$ satisfies Axiom 1′, 2′, 3.2′, 4′, 5′.*

*(2) There exists a unique continuous capacity $c$ on $\mathbb{D}$ such that*

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l, x) \in \mathcal{L}_1 \times X \\ c(D(f,l)) & (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

*represents $\gtrsim'$. For each $f \in \mathcal{F}_a$, $v_f$ defined in Lemma 9 is continuous and strictly decreasing on $v_f^{-1}((0,1))$. Besides, there exists a unique nonempty closed convex set $C \subsetneq \Delta(S)$ such that $c$ is compatible with the information set $C$. Moreover $\mathcal{U}$ is a $\lambda$-system and $\mathcal{U} = \{A \subseteq S \mid p(A) = p'(A)$ for all $p, p' \in C\}$.*

Suppose the hypothesis of Lemma 10 and statement (1). It is easy to check that for any $f \in \mathcal{F}_a$, $\min_{p \in C} E_p u(f) < \max_{p \in C} E_p u(f)$. Moreover,

$$v_f(t) \begin{cases} = 1 & t \leq \min_{p \in C} u(f,p) \\ \in (0,1) & \min_{p \in C} u(f,p) < t < \max_{p \in C} u(f,p) \\ = 0 & t \geq \max_{p \in C} u(f,p) \end{cases}$$

and $v_f$ is strictly decreasing on $(\min_{p \in C} u(f,p), \max_{p \in C} u(f,p))$. For any $f \in \mathcal{F}_0 \backslash \mathcal{F}_a$, $\min_{p \in C} E_p u(f) =$

$\max_{p \in C} E_p u(f)$, and

$$v_f(t) = \begin{cases} 1 & t \leq \min_{p \in C} u(f, p) \\ 0 & t > \min_{p \in C} u(f, p). \end{cases}$$

Let $f \in \mathcal{F}_0$ and $l \in \mathcal{L}_1$ be given. If $E_l u = \min_{p \in C} E_p u(f)$, then we call $l$ an *essential minimum* of f and denote it by $\underline{l}_f$. If $E_l u = \max_{p \in C} E_p u(f)$, then we call $l$ an *essential maximum* of $f$, and denote it by $\bar{l}_f$. We call $[E_{\underline{l}_f} u, E_{\bar{l}_f} u]$ the *essential range* of $f$.

## 3.5 Preference over $\mathcal{F}_0$

We consider the following axioms on the preference over $\mathcal{F}_0$.

**A.2.2. Neutrality.** If $(f, p) \in \mathcal{F}_1$ generates $l_x$ for some $x \in X$, then $(f, p) \sim f_x$.

**A.3.2. Separability.** For any $f \succ g$ in $\mathcal{F}_0$, there exists $l \in \mathcal{L}_1$ such that $f \succ l \succ g$.

**A.4.2. Bound independence.** There exist $\alpha, \beta, \gamma \in (0, 1)$ such that for any $f \in \mathcal{F}_a$, $g \in \mathcal{F}_0 \setminus \mathcal{F}_a$, and $x \prec y$ in $X$,

$$\beta g + (1-\beta) f_x \succsim \alpha f + (1-\alpha) f_x \implies \beta g + (1-\beta) f_y \succsim \alpha f + (1-\alpha) f_y, \tag{6}$$

$$\alpha f + (1-\alpha) f_x \succsim \gamma g + (1-\gamma) f_x \implies \alpha f + (1-\alpha) f_y \succsim \gamma g + (1-\gamma) f_y. \tag{6'}$$

**A.6. Dominance.** For any $f_1, f_2 \in \mathcal{F}_0$, if $(f_1, l) \succsim' (f_2, l)$ for all $l \in \mathcal{L}_1$, then $f_1 \succsim f_2$. For any $N \in \mathbb{Z}_+$, $f, f_1, ..., f_N \in \mathcal{F}_a$, if $\max\{\underline{l}_{f_n} \mid n = 1, ..., N\} \succ \underline{l}_f$ and $\max\{(f_n, l) \mid i = 1, ..., N\} \succ' (f, l)$ for all $l$ such that $\bar{l}_f \succsim l \succ \underline{l}_f$, then $\max\{f_n \mid i = 1, ..., N\} \succ f$.

Axiom A.2.2 assumes that the decision maker is indifferent between a degenerate lottery giving $x \in X$ with probablity 1 and a constant act giving $x$ in each state. Axiom A.3.2

is in the same spirit of the separability axioms in the numerical representation for a preference order (see eg. Debreu (1954)). Axiom A.4.2 is much weaker than the independence axioms in the literature — Gilboa and Schmeidler (1989)'s certainty independence axiom and Maccheroni, Marinacci and Rustichini (2006)'s weak certainty independence axiom.

**B.2. Certainty independence.** For any $f, g \in \mathcal{F}_0$, $x \in X$ and $\alpha \in (0, 1)$,

$$f \succsim g \iff \alpha f + (1-\alpha) f_x \succsim \alpha g + (1-\alpha) f_x. \tag{3.2}$$

Maccheroni, Marinacci, and Rustichini (2006) show that the certainty independence axiom is equivalent to that for any $f, g \in \mathcal{F}_0$, $x, y \in X$ and $\alpha, \beta \in (0, 1]$,

$$\alpha f + (1-\alpha) f_x \succsim \alpha g + (1-\alpha) f_x \implies \beta f + (1-\beta) f_y \succsim \beta g + (1-\beta) f_y. \tag{3.3}$$

This equivalent statement clearly shows that Gilboa and Schmeidler (1989)'s certainty independence involves two types of independence: the independence with respect to mixing with constant acts and the independence with respect to the weights used in such mixing. Maccheroni, Marinacci, and Rustichini (2006) relaxes the second type of independence and propose the weak certainty independence axiom.

**C.2. Weak certainty independence.** For any $f, g \in \mathcal{F}_0$, $x, y \in X$ and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha) f_x \succsim \alpha g + (1-\alpha) f_x \implies \alpha f + (1-\alpha) f_y \succsim \alpha g + (1-\alpha) f_y. \tag{3.4}$$

In both their works and our paper, along with other axioms, each act $f$ is regarded as a state-contingent utility vector $u(f)$ and is evaluated by a functional $I$ on all such utility vectors, i.e., $I(u(f))$. Let $e \in \mathbb{R}^S$ be the unit vector which assigns 1 to each coordinate. For any $f \in \mathcal{F}_0$, $\lambda \in \mathbb{R}_+$ and $t \in \mathbb{R}$, the certainty independence axiom implies that $I(\lambda u(f)+te) = \lambda I(u(f)) + t$, while the weak certainty independence axiom implies that $I(\lambda u(f) + te) = I(\lambda u(f))+t$. The translation invariance of $I$ corresponds to the constant absolute uncertainty aversion property of the preference. Positive homogenerity corresponds to constant relative uncertainty aversion. Thus, B.1 assumes both constant and relative absolute uncertainty aversion, while C.1 retains the absolute part and relaxes the relative part.

Our A.4.2 further relaxes constant absolute uncertainty aversion. It differentiating the effects of a certainty part in different types of acts. Both (6) and (6′) allow the effect of changing the certainty part in an ambiguous act to be different in magnitude from that in an unambiguous act. For example, in (6), an improvement from $f_x$ to $f_y$ by $1-\alpha$ proportion in an ambiguous act may equal to that by $1-\beta$ proportion in an unambiguous act. Moreover, it allows a range of possible effects of improving a certainty part on ambiguous acts rather than a particular one. While (6) implies that an improvement from $f_x$ to $f_y$ by $1-\alpha$ proportion in an ambiguous act will not exceed that by $1-\beta$ proportion in an unambiguous act, (6′) implies that the improvement will not be weaker than that by $1-\gamma$ proportion in an unambiguous act. Under the other axioms, the effects of changing a certainty part in unambiguous acts are normalized, and they are used to measure the effects on ambiguous acts in A.4.2.

From the perspective of the functional form, A.4.2 implies that there exist $k, k' \in \mathbb{R}_+$ such that $I(u(f)+te)-I(u(f)) \in [k't, kt]$ for $f \in \mathcal{F}_a$ and $t \in \mathbb{R}_+$, while $I(u(f)+te) = I(u(f))+t$ for $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$ and $t \in \mathbb{R}$. In A.4.2, if $\beta = \gamma$, then $k = k'$ and $I(u(f) + te) = I(u(f)) + kt$ for $f \in \mathcal{F}_a$ and $t \in \mathbb{R}$. If $\alpha = \beta = \gamma$, then $k = k' = 1$ and $I(u(f) + te) = I(u(f)) + t$ for all $f \in \mathcal{F}_0$ and $t \in \mathbb{R}$, which is the case in Maccheroni, Marinacci, and Rustichini (2006).

The following example shows different behavioral implication of their axioms and ours.

**Example 3.** *Consider an urn containing 90 black and white balls in unknown proportion.*

*A ball is drawn from the urn. The bet $f_t$ pays $50 + t$ dollars whatever happens, while $g_t$ pays $100 + t$ dollars when it is black and t dollars otherwise, where $t \geq 0$. See the following table of payoffs.*

Table 3.1 : Payoffs of $f_t$ and $g_t$

| $t \geq 0$ | Black | White |
|---|---|---|
| $f_t$ | 50+t | 50+t |
| $g_t$ | 100+t | t |

*Clearly, $f_t$ is an unambiguous act for all $t \geq 0$. Since the proportion of black and white balls are unknown, it is reasonable to assume that $g_t$ is ambiguous for the decision maker. Here, t is the ensured payoff by both acts and stands for the wealth level. We further assume for simplicity that the decision maker is risk neutral. Both B.1 and C.1 imply that either $f_t \succsim g_t$ for all t or $g_t \succsim f_t$ for all t. In other words, if the uncertainty is not acceptable when $t = 0$, then it is not acceptable at any wealth level. Neither axiom allows one's uncertainty aversion varies with wealth, while A.4.2 does. Depending on the value of $\alpha, \beta$ and $\gamma$, A.4.2 can accommodates some situations where $f_t \succsim g_t$ for some $t \geq 0$ and $g_t \succsim f_t$ for the other t's. In particular, it may incorporate the situation where there exists $\bar{t}$ such that $f_t \succsim g_t$ when $t \leq \bar{t}$ and $g_t \succsim f_t$ when $t \geq \bar{t}$.*

Lastly, A.6 says that (1) for two acts $f_1, f_2 \in \mathcal{F}_0$, if at any aspiration level, $f_1$ gives weakly more confidence than $f_2$, then $f_1$ is weakly preferred, and (2) for any ambiguous act $f$ and any finite collection of ambiguous acts $f_1, ..., f_N$, $N \in \mathbb{Z}_+$, if the essential minimum of some act in the collection is preferred to that of $f$, and if at any aspiration level, some act in the collection gives more confidence than $f$, then the most preferred act in the collection is preferred to $f$. This axiom is the key for our preference representation.

Let $w : \mathbb{R} \times [0,1] \to [-\infty, \infty)$ be given. We say $w$ is normalized if $w(u,1) = u$ for all $u \in \mathbb{R}$. Let $u : X \to \mathbb{R}$ and $W : \mathcal{F}_0 \to \mathbb{R}$ be given. We say that $W$ is *bounded in translation* if there exists $k, k' > 0$ such that for all $f, g \in \mathcal{F}_a$ and $t > 0$, $u(f) = u(g) + t$ implies $W(f) - W(g) \in [k't, kt]$.

**Theorem 8.** *The following statements are equivalent.*

*(1) $\succsim$ satisfies Axiom 1 - 6, and $\succsim'$ satisfies Axiom 1', 2', 3.2', 4', 5'.*

*(2) (I) There exists a continuous function $u : X \to \mathbb{R}$ unique up to a positive affine transformation, a unique continuous capacity $c$ on $\mathbb{D}$ and a greatest normalized increasing and upper semicontinuous function $w : \mathbb{R} \times [0,1] \to [-\infty, \infty)$ such that*

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l,x) \in \mathcal{L}_1 \times X \\ c(D(f,l)) & (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

*represents $\succsim'$ and*

$$W(\cdot) = \begin{cases} E_l u & l \in \mathcal{L}_1 \\ \max\limits_{t \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]} w(t, v_f(t)) & f \in \mathcal{F}_0 \end{cases}$$

*represents $\succsim$, where $v_f$ is continuous and strictly decreasing on $[E_{\underline{l}_f} u, E_{\bar{l}_f} u]$ when $f \in \mathcal{F}_a$, while W is bounded in translation.*

*(II) There exists a unique non empty closed convex set $C \subseteq \Delta(S)$ such that $c$ is compatible with the information set $C$. Moreover, $\mathcal{U}$ is a $\lambda$-system and $\mathcal{U} = \{A \subseteq S \mid p(A) = p'(A)$ for all $p,\ p' \in C\}$.*

The interpretation is that the decision maker treats every act as a trade-off between aspiration and confidence, and he evaluates the act by the optimal combination of aspiration and confidence level according to an aggregating preference on the aspiration and confidence two-criterion plane. It resembles the structure in the standard consumer theory. A consumer evaluates a budget set on the two-commodity space by the optimal bundle as determined by his preference over the commodity space.

One can show that if $w(u,t) = -\infty$ for some $(u,t) \in \mathbb{R} \times [0,1]$, then $w(u',t) = -\infty$ for all $u' \in \mathbb{R}$. This means that the decision maker has a threshold value of confidence level, below which he regards it as unacceptable. The maximin expected utility (MEU) in Gilboa and Schmeidler (1989) is an example. A MEU decision maker evaluates an act by its minimum expected utility among a nonempty convex closed set of priors. This set is interpreted as his information set in our framework, and his aggregating preference is

$$w(u,t) = \begin{cases} u & t = 1 \\ -\infty & t < 1. \end{cases}$$

Hence, a MEU decision maker only consider the aspiration levels that corresponds the full full confidence, among which the highest level is the minimum expected utility over $C$.

## 3.6 Appendix

Before proving Lemma 8, we provide an analogy of the classic expected utility representation result with the preference $\succsim$ defined on $\mathcal{L}_1 = \{l \in \mathcal{L}_0 \mid |\operatorname{supp}(l)| \leq |S|\}$. The following axioms are standard except that here they apply to $\succsim$ when the relevant alternatives exist in $\mathcal{L}_1$.

**B.1. Weak Order.** $\succsim$ is complete and transitive.

**B.2. Independence.** For any $l_1, l_2, l_3 \in \mathcal{L}_1$ and $\lambda \in (0,1)$, $l_1 \succ l_2$ implies that $l_1 \lambda l_3 \succ l_2 \lambda l_3$ if both $l_1 \lambda l_3$ and $l_2 \lambda l_3$ exist in $\mathcal{L}_1$.

**B.3. Continuity.** For any $l_1 \succ l_2 \succ l_3$ in $\mathcal{L}_1$, if $l_1\lambda l_3$ exists for some $\lambda \in (0,1)$, then $l_1\alpha l_3 \succ l_2$ and $l_2 \succ l_1\beta l_3$ for some $\alpha, \beta \in (0,1)$.

**Lemma 11.** *A preference $\succsim$ satisfies B.1, B.2 and B.3 if and only if there is a function $u : X \to \mathbb{R}$ such that $l \succsim l' \Leftrightarrow E_l u \geq E_{l'} u$ for all $l, l' \in \mathcal{L}_1$. Moreover, $u$ is unique up to a positive affine transformation.*

*Proof.* The necessity is obvious. We show the sufficiency in three steps. The proof is based on that of Theorem 8.3 and 8.4 in Fishburn (1970). The only difference here is to check which properties hold without the "mixture set" assumption and whether they are sufficient to derive an expected utility representation.

**Step 1.** If $\succsim$ satisfies B.1, B.2 and B.3, then the following holds for all $l_1, l_2, l_3 \in \mathcal{L}_1$.

**C.1.** Suppose that $l_1 \succ l_2$ and $0 \leq \alpha < \beta \leq 1$. Then $l_1\beta l_2 \succ l_1\alpha l_2$ if they exist in $\mathcal{L}_1$.

**C.2.** Suppose that $l_1 \succsim l_2$, $l_2 \succsim l_3$ and $l_1 \succ l_3$. If $l_1\lambda l_3$ exists for some $\lambda \in (0,1)$, then $l_2 \sim l_1\alpha l_3$ for exactly one $\alpha \in [0,1]$.

**C.3.** Suppose that $l_1 \sim l_2$ and $0 \leq \alpha \leq 1$. Then $l_1\alpha l_2 \sim l_1$ if $l_1\alpha l_2$ exists in $\mathcal{L}_1$.

**C.4.** Suppose that $l_1 \sim l_2$ and $0 \leq \alpha \leq 1$. Then $l_1\alpha l_3 \sim l_2\alpha l_3$ if they exist in $\mathcal{L}_1$.

The proof of C.1 and C.2 is exactly the same as Fishburn's proof. To check C.3, the case when $\alpha = 0$ or 1 is easy. Suppose that $0 < \alpha < 1$, $l_1\alpha l_2$ exists in $\mathcal{L}_1$ and $l_1\alpha l_2 \succ l_1$. Then by B.2, we have $(l_1\alpha l_2)\alpha l_2 \succ l_1\alpha l_2$. Since $l_1\alpha l_2 \succ l_2$, by C.1, $l_1\alpha l_2 \succ (l_1\alpha l_2)\alpha l_2$ which is a contradiction. The case when $l_1 \succ l_1\alpha l_2$ can lead to a similar contradiction. Hence, $l_1\alpha l_2 \sim l_1$.

For C.4, it holds obviously when $\alpha = 0$ or 1, or $l_3 \sim l_1$. Suppose that $0 < \alpha < 1$, $l_1\alpha l_3$ and $l_2\alpha l_3$ exist in $\mathcal{L}_1$, and $l_3 \succ l_1$. Then by C.1, $l_3 \succ l_1\alpha l_3 \succ l_1$. Thus $l_3 \succ l_1\alpha l_3 \succ l_2$, and by C.1, $l_1\alpha l_3 \sim l_2\beta l_3$ for some $\beta \in [0,1]$. Suppose that $\beta < \alpha$. By C.1, $l_2\frac{\beta}{\alpha}l_3 \succ l_2 \sim l_1$. By B.2, $(l_2\frac{\beta}{\alpha}l_3)\alpha l_3 \succ l_1\alpha l_3$. Then $l_2\beta l_3 = (l_2\frac{\beta}{\alpha}l_3)\alpha l_3 \succ l_1\alpha l_3$ which is a contradiction. Similarly, it cannot be that $\beta > \alpha$. Thus, $l_1\alpha l_3 \sim l_2\alpha l_3$.

**Step 2.** Assume that $l_x \succ l_y$ for some $x, y \in X$. Let $l_xl_y = \{l \in \mathcal{L}_1 \mid l_x \succsim l \succsim l_y\}$. Then there exists a function $f : l_xl_y \to [0,1]$ such that for all $l, l' \in l_xl_y$, (1) $l \succsim l'$ if and only if $f(l) \geq f(l')$, and (2) for all $\alpha \in [0,1]$, $f(l\alpha l') = \alpha f(l) + (1-\alpha)f(l')$ if $l\alpha l'$ exists in $\mathcal{L}_1$.

For all $l \in l_xl_y$, let $f(l)$ to be the unique number in $[0,1]$ such that $l \sim l_xf(l)l_y$. The function $f$ is well-defined by C.2. By C.1, $l_xf(l)l_y \succsim l_xf(l')l_y$ if and only if $f(l) \geq f(l')$. Thus, (1) holds by the definition of $f$.

To check (2), the case when $\alpha = 0$ or $1$ is obvious. Suppose that $0 < \alpha < 1$ and $l\alpha l'$ exists in $\mathcal{L}_1$. Let $z, w \in \text{supp}(l) \cup \text{supp}(l')$ be given such that $l_z \succsim l_r \succsim l_w$ for all $r \in \text{supp}(l) \cup \text{supp}(l')$. By repeatedly using C.1 or C.3, we have $l_z \succsim l \succsim l_w$. Thus by C.2 or C.3, $l \sim l_z\beta l_w$ for some $\beta \in [0,1]$. Similarly, $l' \sim l_z\gamma l_w$ for some $\gamma \in [0,1]$. Note that $(l_z\beta l_w)\alpha l'$ also exists in $\mathcal{L}_1$. By C.4, $l\alpha l' \sim (l_z\beta l_w)\alpha l'$. Similarly, $(l_z\beta l_w)\alpha l' \sim (l_z\beta l_w)\alpha(l_z\gamma l_w)$. Hence, $l\alpha l' \sim (l_z\beta l_w)\alpha(l_z\gamma l_w)$.

If $l \sim l'$, then (2) holds trivially. Suppose without loss of generality that $l \succ l'$. Then $l_z \succ l_w$, $\beta > \gamma$ and $f(l) > f(l')$. Note that $l_x \succsim l \sim l_z\beta l_w \succ l_w$. By C.2, $l_z\beta l_w \sim l_x\beta' l_w$ for a unique $\beta' \in [0,1]$. Similarly, $l_z\gamma l_w \sim l_x\gamma' l_w$ for a unique $\gamma' \in [0,1]$, and $\beta' > \gamma'$. Note that $l_z\gamma l_w = (l_z\beta l_w)\frac{\gamma}{\beta}l_w \sim (l_x\beta' l_w)\frac{\gamma}{\beta}l_w = l_x\frac{\beta'\gamma}{\beta}l_w$ by C.4. Then the uniqueness of $\gamma'$ implies that $\gamma' = \frac{\beta'\gamma}{\beta}$. Hence, $(l_z\beta l_w)\alpha(l_z\gamma l_w) = (l_z\beta l_w)\alpha[(l_z\beta l_w)\frac{\gamma}{\beta}l_w] = (l_z\beta l_w)[\alpha + (1-\alpha)\frac{\gamma}{\beta}]l_w \sim (l_x\beta' l_w)[\alpha + (1-\alpha)\frac{\gamma}{\beta}]l_w = (l_x\beta' l_w)[\alpha + (1-\alpha)\frac{\gamma'}{\beta'}]l_w = (l_x\beta' l_w)\alpha(l_x\gamma' l_w)$.

Analogously, $l_x\beta' l_w \sim l_xf(l)l_y$ and $l_x\gamma' l_w \sim l_xf(l')l_y$. Since $l_x\beta' l_w = l_x\frac{\beta'-\gamma'}{1-\gamma'}(l_x\gamma' l_w) \sim l_x\frac{\beta'-\gamma'}{1-\gamma'}[l_xf(l')l_y]$, $l_xf(l)l_y = l_x\frac{f(l)-f(l')}{1-f(l')}[l_xf(l')l_y]$ and $l_x \succ l_xf(l')l_y$, then $\frac{\beta'-\gamma'}{1-\gamma'} = \frac{f(l)-f(l')}{1-f(l')}$ by C.1. Hence, $(l_x\beta' l_w)\alpha(l_x\gamma' l_w) = l_x\frac{\alpha(\beta'-\gamma')}{1-\gamma'}(l_x\gamma' l_w) \sim l_x\frac{\alpha(\beta'-\gamma')}{1-\gamma'}[l_x f(l')l_y] = l_x\frac{\alpha[f(l)-f(l')]}{1-f(l')}[l_x f(l')l_y] = [l_x f(l)l_y]\alpha[l_x f(l')l_y]$. Therefore, we get that $l\alpha l' \sim [l_x f(l)l_y]\alpha[l_x f(l')l_y] = l_x[\alpha f(l) + (1-\alpha)f(l')]l_y$, and thus by the definition of $f$, $f(l\alpha l') = \alpha f(l) + (1-\alpha)f(l')$.

**Step 3.** There exsits a function $u : X \to \mathbb{R}$ such that $l \gtrsim l' \Leftrightarrow E_l u \geq E_{l'} u$ for all $l, l' \in \mathcal{L}_1$.

If there are no $x, y \in X$ such that $l_x \succ l_y$, $l \sim l'$ for all $l, l' \in \mathcal{L}_1$ by repeatedly using C.3. Thus any constant $u$ works. Suppose that there exists $l_x \succ l_y$ for some $x, y \in X$. For $i = 1$ or 2, let $l_{x_i}l_{y_i} = \{l \in \mathcal{L}_1 \mid l_{x_i} \gtrsim l \gtrsim l_{y_i}\}$ such that $l_x l_y \subseteq l_{x_i}l_{y_i}$. For both $i$, let $f_i' : l_{x_i}l_{y_i} \to [0,1]$ be the function constructed as in Step 2, let $f_i$ be the affine transformation of $f_i'$ such that $f_i(l_x) = 1$ and $f_i(l_y) = 0$, and thus $f_i$ still satisfies (1) and (2) in Step 2.

Let $l \in l_{x_1}l_{y_1} \cap l_{x_2}l_{y_2}$. If $l \sim l_x$ or $l \sim l_y$, then $f_1(l) = f_2(l)$. Otherwise, one of the following cases must be true: $l_{x_i} \gtrsim l \succ l_x \succ l_y$, $l_x \succ l \succ l_y$ or $l_x \succ l_y \succ l \gtrsim l_{y_i}$. Consider the first case. Suppose without loss of generality that $l_{x_1} \gtrsim l_{x_2}$. Then for both $i$, $l_{x_2} \in l_{x_i}l_{y_i}$, and $l_x \sim l_{x_2}\alpha l_y$ for a unique $\alpha \in (0,1)$ by C.2. Hence, $1 = f_i(l_x) = \alpha f_i(l_{x_2}) + (1-\alpha)f_i(l_y) = \alpha f_i(l_{x_2})$ for $i = 1, 2$. Since $l \sim l_{x_2}\beta l_y$ for a unique $\beta \in (0,1]$, then $f_i(l) = \beta f_i(l_{x_2}) = \frac{\beta}{\alpha}$ for $i = 1, 2$. Similarly, in the other cases, we also get $f_1(l) = f_2(l)$.

For all $l \in \mathcal{L}_1$, let $f(l)$ be the common value of $f_i(l)$ defined on every such $l_{x_i}l_{y_i}$ as above. Since each pair of $l, l' \in \mathcal{L}_1$ is contrained in some $l_{x_i}l_{y_i}$, then $f$ satisfies condition (1) and (2) in Step 2. Define $u : X \to \mathbb{R}$ by $u(x) = f(l_x)$ for all $x \in X$. Finally, for all $l \in \mathcal{L}_1$, $f(l) = f[\sum_{x\in\text{supp}(l)} l(x)l_x] = \sum_{x\in\text{supp}(l)} l(x)f(l_x) = \sum_{x\in\text{supp}(l)} l(x)u(x) = E_l u$.

We complete the proof by checking the uniqueness property. Let $v : X \to \mathbb{R}$ be such that $l \gtrsim l' \Leftrightarrow E_l v \geq E_{l'} v$ for all $l, l' \in \mathcal{L}_1$. If there is no $l_x \succ l_y$ in $X$, then $u$ and $v$ constant on $X$. Clearly, $u$ is an affine transformation of $v$. Otherwise, fix some $x, y \in X$ such that $l_x \succ l_y$. Let $u'(z) = \frac{u(z)-u(y)}{u(x)-u(y)}$ and $v'(z) = \frac{v(z)-v(y)}{v(x)-v(y)}$ for all $z \in X$. Note that $u'$ and $v'$ are affine transformations of $u$ and $v$, so for all $l \in \mathcal{L}_1$, $l \succsim l' \Leftrightarrow E_l u' \geq E_{l'} u' \Leftrightarrow E_l v' \geq E_{l'} v'$. Besides, $u'(x) = v'(x) = 1$ and $u'(y) = v'(y) = 0$. Fix $z \in X$. Then one of the following cases is true: $l_z \succsim l_x \succ l_y$, $l_x \succ l_z \succ l_y$ or $l_x \succ l_y \succsim l_z$. Using the similar argument as before, we get $u'(z) = v'(z)$ in all cases. Thus $v(z) = \frac{v(x)-v(y)}{u(x)-u(y)} u(z) + v(y) - \frac{v(x)-v(y)}{u(x)-u(y)} u(y)$ for all $z \in X$. This shows that $v$ is an affine transformation of $u$. $\square$

Using Lemma 11, we can prove Lemma 8. Let $\mathbb{Z}_+$ denote the set of positive integers.

*Proof of Lemma 8.* The necessity part is easy. Let us check the sufficiency. By A.2.1, we only need to show the analogous representation result for $\succsim$ on $\mathcal{L}_1$.

Clearly A.1 implies B.1. We show that A.3.1 implies B.3. Let $l_1 \succ l_2 \succ l_3$ in $\mathcal{L}_1$ be given and suppose that ł$_1 \lambda l_3$ exists for some $\lambda \in (0, 1)$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of elements in $(0, 1)$ such that $\lim\limits_{n\to\infty} \lambda_n = 1$. For any continuous real-valued function $\eta$ on $X$, $\lim\limits_{n\to\infty} \int \eta d(l_1 \lambda_n l_3) = \int \eta dl_1$. Hence, $\lim\limits_{n\to\infty} l_1 \lambda_n l_3 = l_1$. Since $l_1 \succ l_2$ and $\{l \in \mathcal{L}_1 \mid l \succ l_2\}$ is open by A.3.1, then there exists $N \in \mathbb{Z}_+$ such that $l_1 \lambda_N l_3 \succ l_2$. Similarly, pick a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of elements in $(0, 1)$ converging to 0, then there is $N' \in \mathbb{Z}_+$ such that $l_2 \succ l_1 \lambda_N l_3$.

By applying Lemma 11, there exists a function $u : X \to \mathbb{R}$ such that $l \succsim l' \Leftrightarrow E_l u \geq E_{l'} u$ for all $l, l' \in \mathcal{L}_1$. Moreover, $u$ is unique up to a positive affine transformation. To show that $u$ is continuous, suppose the contrary that there exist $\epsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in $X$ such that $\lim\limits_{n\to\infty} x_n = x_0$, $x_0 \in X$, and $|u(x_n) - u(x_0)| > \epsilon$ for all $n$. Then there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that either $u(x_{n_j}) > u(x_0) + \epsilon$ for all $j$ or $u(x_{n_j}) < u(x_0) - \epsilon$ for all $j$. Assume the former case and let $\underline{u} = \inf\{u(x_{n_j}) \mid j \in \mathbb{Z}_+\}$. Pick $J \in \mathbb{Z}_+$ such that $u(x_{n_J}) < \underline{u} + \epsilon$. Let $l = l_{x_0} \frac{1}{2} l_{x_{n_J}}$. For all $j \in \mathbb{Z}_+$, $u(x_{n_j}) \geq \underline{u}$ and $\frac{1}{2} u(x_0) + \frac{1}{2} u(x_{n_J}) < \frac{1}{2}(\underline{u} - \epsilon) + \frac{1}{2}(\underline{u} + \epsilon) = \underline{u}$, so $l_{x_{n_j}} \succsim l$. For any continuous real-valued function $\eta$ on $X$,

$\lim_{j\to\infty}\int \eta dl_{x_{n_j}} = \lim_{j\to\infty}\eta(x_{n_j}) = \eta(x_0) = \int \eta dl_{x_0}$. Thus $\lim_{j\to\infty} l_{x_{n_j}} = l_{x_0}$. By A.3, $l_{x_0} \gtrsim l$, which contradicts that $u(x_0) < \frac{1}{2}u(x_0) + \frac{1}{2}\underline{u} \leq \frac{1}{2}u(x_0) + \frac{1}{2}u(x_{n_J})$. The argument follows analogously for the case when $u(x_{n_j}) < u(x_0) - \epsilon$ for all $j$.

Lastly, if $u(X) = \mathbb{R}$, then A.5 obviously holds. For the other direction, assume A.5 holds. Then we have $l_x \succ l_y$ such that for all $\lambda \in (0,1)$, there exists $z_1 \in X$ such that $u(y) > \lambda u(z_1) + (1-\lambda)u(x)$, i.e., $u(z_1) < \frac{1}{\lambda}[u(y) - (1-\lambda)u(x)]$. Thus, $\lim_{\lambda\to 0} u(z_1) = -\infty$. Similarly, $u(X)$ is not bounded above. Since $u$ is continuous, $X$ is connected and $u(X)$ is unbounded, then $u(X) = \mathbb{R}$.

□

*Proof of Lemma 9.* Suppose that (1) holds. Fix $x_0, x_1 \in X$ such that $l_{x_1} \succ l_{x_0}$.

**Step 1.** If $l_y \gtrsim l_x$ ($l_x \succ l_y$) for all $y \in \text{supp}(l)$, then $(l,x) \in \max \mathcal{M}$ ($(l,x) \in \min \mathcal{M}$). If $(f,p) \gtrsim l$ ($l \succ (f,p)$) for all $p \in \Delta(S)$, then $(f,l) \in \max \mathcal{M}$ ($(f,l) \in \min \mathcal{M}$).

By A.3.1′, if $l_y \gtrsim l_x$ ($l_x \succ l_y$) for all $y \in \text{supp}(l)$, then $(l,x) \in \max \mathcal{L}_1 \times X$ ($(l,x) \in \min \mathcal{L}_1 \times X$), and if $(f,p) \gtrsim l$ ($l \succ (f,p)$) for all $p \in \Delta(S)$, then $(f,l) \in \max \mathcal{F}_0 \times \mathcal{L}_1$ ($(f,l) \in \min \mathcal{F}_0 \times \mathcal{L}_1$). In particular, $(f_{x_0}, l_{x_0}) \in \max \mathcal{F}_0 \times \mathcal{L}_1$, $(f_{x_0}, l_{x_1}) \in \min \mathcal{F}_0 \times \mathcal{L}_1$, $(l_{x_0}, x_0) \in \max \mathcal{L}_1 \times X$, and $(l_{x_0}, x_1) \in \min \mathcal{L}_1 \times X$. Combining these facts and A.4’, we get that the desired results.

**Step 2.** For any $(f,l) \in \mathcal{F}_0 \times \mathcal{L}_1$, there is a unique $\lambda \in [0,1]$ such that $(f,l) \sim' (l_{x_1}\lambda l_{x_0}, x_1)$.

Note that $(l_{x_1}\alpha l_{x_0}, x_1) \succ' (l_{x_1}\beta l_{x_0}, x_1)$ if and only if $\alpha > \beta$ in $[0,1]$. Moreover, $(l_{x_0}, x_1) \in \min \mathcal{M}$ and $(l_{x_1}, x_1) \in \max \mathcal{M}$. Hence, if $(f,l) \in \min \mathcal{M} \cup \max \mathcal{M}$, then the unique $\lambda$ is either 0 or 1. On the other hand, if $(l_{x_1}, x_1) \succ' (f,l) \succ' (l_{x_0}, x_1)$, then by $A.2'(1)$

there exists a unique $\lambda \in (0,1)$ such that for all $\alpha, \beta$ $(0,1)$ with $1 \geq \alpha > \lambda > \beta \geq 0$, $(l_{x_1}\alpha l_{x_0}, x_1) \succ' (f,l) \succ' (l_{x_1}\beta l_{x_0}, x_1)$. If $(l_{x_1}\lambda l_{x_0}, x_1) \succ (f,l)$, then again by $A.2'$, there exists $\mu \in (0,1)$ such that $(l_{x_1}\lambda\mu l_{x_0}, x_1) \succ' (f,l)$, which is a contradiction since $\lambda\mu < \lambda$. Similarly, it cannot be true that $(f,l) \succ' (l_{x_1}\lambda l_{x_0}, x_1)$. Hence, $(f,l) \sim' (l_{x_1}\lambda l_{x_0}, x_1)$.

**Step 3.** For all $(f,l) \in \mathcal{F}_0 \times \mathcal{L}_1$, define $c(D(f,l)) = \lambda$ if $(f,l) \sim' (l_{x_1}\lambda l_{x_0}, x_1)$, $\lambda \in [0,1]$. Clearly, $c : \mathbb{D} \to \mathbb{R}$ is well defined and it is the unique function on $\mathbb{D}$ such that

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l,x) \in \mathcal{L}_1 \times X \\ c(D(f,l)) & (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

represents $\succsim'$ on $\mathcal{M}$. We would like to show that $c$ is a continuous capacity on $\mathbb{D}$.

First, $c$ is a capacity on $\mathbb{D}$. Note that $c(\emptyset) = c(D(f_{x_0}, l_{x_1})) = 0$ since $(f_{x_0}, l_{x_1}) \sim' (l_{x_0}, x_1)$, and $c(\Delta(S)) = c(D(f_{x_1}, l_{x_1})) = 1$ since $(f_{x_1}, l_{x_1}) \sim' (l_{x_1}, x_1)$. If $D(f,l) \supseteq D(f',l')$, then $(f,l) \succsim' (f',l')$. Suppose that $(f,l) \sim' (l_{x_1}\lambda l_{x_0}, x_1)$ and $(f',l') \sim' (l_{x_1}\lambda' l_{x_0}, x_1)$. Hence, $\lambda \geq \lambda'$ and thus $c(D(f,l)) \geq c(D(f',l'))$.

To check $c$ is upper continuous, let $\{D(f_n,l_n)\}_{n=1}^{\infty}$ be a non-increasing sequence of sets in $\mathbb{D}$ and $\cap_{n=1}^{\infty} D(f_n,l_n) = D(f,l) \in \mathbb{D}$. We want to show that $\lim_{n\to\infty} c(D(f_n,l_n)) = c(D(f,l))$. Since $c$ is monotone and bounded, then $\underline{c} := \lim_{n\to\infty} c(D(f_n,l_n)) = \inf\{c(D(f_n,l_n)) \mid n \in \mathbb{Z}_+\} \in [0,1]$. Note that $\underline{c} \geq c(D(f,l))$. If $\underline{c} = 0$, then $\underline{c} = c(D(f,l)) = 0$. Suppose that $\underline{c} > 0$. Thus, $D(f_n,l_n) \neq \emptyset$ for each $n$. Pick $p_n \in \arg\min_{p \in D(f_n,l_n)} E_p u(f)$ and let $l'_n = (f, p_n)$ for each $n$. Clearly, $l'_n \succsim l'_m$ when $n \geq m$, which means that $D(f,l'_n) \subseteq D(f,l'_m)$ when $n \geq m$. Moreover, $D(f_n,l_n) \subseteq D(f,l'_n)$, since $p \in D(f_n,l_n)$ implies that $E_p u(f) \geq E_{p_n} u(f) = E_{l_n} u$ and thus $(f,p) \succsim l_n$. Hence, $D(f,l) \subseteq \cap_{n=1}^{\infty} D(f,l'_n)$.

In fact, $D(f,l) = \cap_{n=1}^{\infty} D(f,l'_n)$. Suppose the contrary that there exists $p' \in \cap_{n=1}^{\infty} D(f,l'_n) \setminus$

$D(f,l)$. Thus $l \succ (f,p')$. Pick $l' \in \mathcal{L}_1$ be such that $l \succ l' \succ (f,p')$. Thus $p' \notin D(f,l')$. We check that $D(f_n,l_n) \subseteq D(f,l')$ for some and thus for all sufficiently large $n \in \mathbb{Z}_+$. Suppose that there exists $p'_n \in D(f_n,l_n) \setminus D(f,l')$ for each $n$. The sequence $\{p'_n\}_{n=1}^{\infty}$ is bounded, so there is a subsequence $\{p'_{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} p'_{n_j} = p^* \in \Delta(S)$. For each $n \in \mathbb{Z}_+$, $p'_{n_j} \in D(f_{n_j},l_{n_j}) \subseteq D(f_n,l_n)$ for sufficiently large $j$, and since $D(f_n,l_n)$ is closed, then $p^* \in D(f_n,l_n)$. Thus $p^* \in \cap_{n=1}^{\infty} D(f_n,l_n)$. Since $p'_n \notin D(f,l')$ for each $n$ and $l \succ l'$, then $E_{p^*}u(f) = \lim_{j\to\infty} E_{p'_{n_j}}u(f) \leq E_{l'}u < E_l u$, and thus $p^* \notin D(f,l)$. This is a contradiction to $p^* \in \cap_{n=1}^{\infty} D(f_n,l_n) = D(f,l)$. Hence, $D(f_n,l_n) \subseteq D(f,l')$ for sufficiently large $n$. Next, note that if $D(f_n,l_n) \subseteq D(f,l')$, then $E_{l'_n}u = \min_{p\in D(f_n,l_n)} E_p u(f) \geq E_{l'}u$, and thus $l'_n \gtrsim l'$. Combining the facts above, we have that $D(f,l'_n) \subseteq D(f,l')$ for sufficiently large $n \in \mathbb{Z}_+$. Since $p' \in \cap_{n=1}^{\infty} D(f,l'_n)$, then $p' \in D(f,l')$, which contradicts with $l' \succ (f,p')$.

Let $\underline{c}' := \inf\{c(D(f,l'_n)) \mid n \in \mathbb{Z}_+\} = \lim_{n\to\infty} c(D(f,l'_n))$. Clearly, $\underline{c}' \in [0,1]$. For each $n \in \mathbb{Z}_+$, by the definition of $l'_n$, $D(f_n,l_n) \subseteq D(f,l'_n)$, so $\underline{c} \leq \underline{c}'$. Since $c(D(f,l)) \leq \underline{c}$, then it suffices to show that $\underline{c}' = c(D(f,l))$.

Suppose that $D(f,l) = \emptyset$. We want to show that $D(f,l'_n) = \emptyset$ for sufficiently large $n$ and thus $\underline{c}' = c(D(f,l)) = 0$. To show that, assume the opposite that for all $n \in \mathbb{Z}_+$, $D(f,l'_n) \neq \emptyset$ and thus $E_{l'_n}u \leq \max_{s\in S} u(f(s))$. Let $\underline{u} = \sup\{E_{l'_n}u \mid n \in \mathbb{Z}_+\}$. Then $\underline{u} \in \mathbb{R}$ and there exists $\underline{l} \in \mathcal{L}_1$ such that $E_{\underline{l}}u = \underline{u}$. Moreover, $D(f,l) = \cap_{n=1}^{\infty} D(f,l'_n) = D(f,\underline{l})$. For each $n \in \mathbb{Z}_+$, choose $p_n \in D(f,l'_n)$. Let $\{p_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{p_n\}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} p_{n_j} = \underline{p} \in \Delta(S)$. Hence, $E_{\underline{p}}u(f) = \lim_{j\to\infty} E_{p_{n_j}}u(f) \geq \lim_{j\to\infty} E_{l'_{n_j}}u = \underline{u}$, and thus $\underline{p} \in D(f,\underline{l}) = D(f,l)$. This contradicts with $D(f,l) = \emptyset$.

Next, suppose that $\max_{s\in S} u(f(s)) = \min_{s\in S} u(f(s))$. Then $D(f,l)$ and $D(f,l'_n)$, $n \in \mathbb{Z}_+$, are either  or $\Delta(S)$. Hence, if $D(f,l) = \emptyset$, then $D(f,l'_n) = \emptyset$ for sufficiently large $n$; if $D(f,l) = \Delta(S)$, then $D(f,l'_n) = \Delta(S)$ for all $n \in \mathbb{Z}_+$. In either case, $\underline{c}' = c(D(f,l))$.

Lastly, suppose that $D(f,l) \neq \emptyset$, $\max_{s\in S} u(f(s)) \neq \min_{s\in S} u(f(s))$, and $\underline{c}' > c(D(f,l))$. Then

$D(f,l) \neq \Delta(S)$ as well. Choose $\lambda \in (c(D(f,l)), \underline{c}')$. Then $(f, l'_n) \succ' (l_{x_1}\lambda l_{x_0}, x_1) \succ' (f,l)$ for all $n \in \mathbb{Z}_+$. By $A.2'(2)$, for each $n \in \mathbb{Z}_+$, there exists $\beta_n \in (0,1)$ such that $(l_{x_1}\lambda l_{x_0}, x_1) \succ' (f, l'_n\beta_n l)$. Fix any $N \in \mathbb{Z}_+$. If there exists $m_N \in \mathbb{Z}_+$ such that $l'_{m_N} \succsim l'_N\beta_N l$, then $D(f, l'_{m_N}) \subseteq D(f, l'_N\beta_N l)$, and thus $c(D(f, l'_{m_N})) \leq c(D(f, l_N\beta_N l)) < \lambda < \underline{c}$ which is a contradiction. To see that $l'_{m_N} \succsim l'_N\beta_N l$ for some $m_N \in \mathbb{Z}_+$, suppose the contrary that $l'_N\beta_N l \succ l'_m$ for all $m \in \mathbb{Z}_+$. Hence, $D(f, l'_N\beta_N l) \subseteq D(f, l'_m)$ for all $m \in \mathbb{Z}_+$, and then $D(f, l'_N\beta_N l) \subseteq D(f,l)$. On the other hand, since $(f, l'_N) \succ' (f,l)$, then $l \succ l'_N$ and thus $l \succ l'_N\beta_N l$. Since $D(f,l)$ is neither $\emptyset$ nor $\Delta(S)$, and $\max_{s\in S} u(f(s)) \neq \min_{s\in S} u(f(s))$, then $E_l u \in (\min_{s\in S} u(f(s)), \max_{s\in S} u(f(s))]$. Observe that there must exist $q \in \Delta(S)$ such that $E_{l'_N\beta_N l} u < E_q u(f) < E_l u$. Thus, $q \in D(f, l'_N\beta_N l) \setminus D(f,l)$ which is a contradiction, as desired.

To check that $c$ is lower continuous, let $\{D(f_n, l_n)\}_{n=1}^{\infty}$ be a non-decreasing sequence of sets in $\mathbb{D}$ and $\cup_{n=1}^{\infty} D(f_n, l_n) = D(f,l)$. Note that $D(f,l)$ is the intersection of finitely many half spaces, that is, it is a convex polytope. By the vertex representation of a convex polytope, it can be written as the convex hull of finitely many points of it. Since $\{D(f_n, l_n)\}_{n=1}^{\infty}$ is a non-decreasing sequence of convex sets, then $D(f,l) \subseteq D(f_n, l_n)$ for sufficiently large $n$. This implies that $D(f,l) = D(f_n, l_n)$ for sufficiently large $n$. Therefore, $\lim_{n\to\infty} c(D(f_n, l_n)) = c(D(f,l))$.

**Step 4.** For each $f \in \mathcal{F}_0$, the robustness index $v_f : \mathbb{R} \to [0,1]$ defined by $v_f(t) = V(f,l)$ if $E_p u = t$ is continuous when $v_f(\mathbb{R}) \neq \{0,1\}$.

Fix $f \in \mathcal{F}_0$. We first check that $v_f$ is well-defined. For any $t \in \mathbb{R}$, there exists $l \in \mathcal{L}_1$ such that $E_l u = t$, since $u(X) = \mathbb{R}$. Suppose that $l' \in \mathcal{L}_1$, $l' \neq l$ and $E'_l u = t$. Hence, $D(f,l) = D(f,l')$ and thus $V(f,l) = V(f,l')$. Note that $v_f$ is non-increasing, $v_f(t) = 1$ when $t \leq \min_{s\in S} u(f(s))$, and $v_f(t) = 0$ when $t > \max_{s\in S} u(f(s))$.

Next, suppose that $v_f(\mathbb{R}) \neq \{0,1\}$, and we would like to show that $v_f$ is continuous. In this case, it must be true that $\max_{s\in S} u(f(s)) \neq \min_{s\in S} u(f(s))$. Suppose the opposite that $v_f$ is not continous at $t \in \mathbb{R}$. To begin with, we assume that $v_f(t) \in (0,1)$. Then there exist $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^{\infty}$ of elements in $\mathbb{R}$ such that $\lim_{n\to\infty} t_n = t$ and $|v_f(t) - v_f(t_n)| \geq \epsilon$. Thus there is a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$ such that $t_{n_j} > t$ for all $j \in \mathbb{Z}_+$ or $t_{n_j} < t$ for all $j \in \mathbb{Z}_+$. Suppose the former case. Then $v_f(t_{n_j}) \leq v_f(t) - \epsilon$ for all $j \in \mathbb{Z}_+$. Let $l \in \mathcal{L}_1$ and $l_{n_j} \in \mathcal{L}_1$ for each $j \in \mathbb{Z}_+$ be such that $E_l u = t$ and $E_{l_{n_j}} u = t_{n_j}$. Let $\lambda = v_f(t) - \frac{2}{\epsilon}$. Then $(f, l) \succ' (l_{x_1} \lambda l_{x_2}, x_1) \succ' (f, l_{n_j})$ for all $j \in \mathbb{Z}_+$. By $A.2'(2)$, there exists $\alpha_j \in (0,1)$ for each $j \in \mathbb{Z}_+$ such that $(f, l\alpha_j l_{n_j}) \succ' (l_{x_1}\lambda l_{x_2}, x_1)$. Note also that $l > l_{n_j}$ and thus $l\alpha_j l_{n_j} > l$, $j \in \mathbb{Z}_+$. Fix $J \in \mathbb{Z}_+$. Since $\lim_{j\to\infty} t_{n_j} = t$, then there exists $K \in \mathbb{Z}_+$ such that $l\alpha_J l_{n_J} > l_{n_K} > l$. Hence, $(f, l_{n_K}) \succsim' (f, l\alpha_J l_{n_J})$ and thus $(f, l_{n_K}) \succ' (l_{x_1}\lambda l_{x_2}, x_1)$, which contradicts that $(l_{x_1}\lambda l_{x_2}, x_1) \succ' (f, l_{n_j})$ for all $j \in \mathbb{Z}_+$. Suppose the later case where for all $j \in \mathbb{Z}_+$, $t_{n_j} < t$ so that $v_f(t_{n_j}) \geq v_f(t) + \epsilon$. A similar argument as above lead to another contradiction. If $v_f(t) = 1$, then $v_f(t') = 1$ for all $t' \leq t$. Thus $t_n > t$ for all $n \in \mathbb{Z}_+$. Since $v_f(\mathbb{R}) \neq \{0,1\}$, then there exists $N \in \mathbb{Z}_+$ such that $v_f(t_N) > 0$, and thus $(f, l_N) \notin \min \mathcal{M}$. Then $A.2'(2)$ can be used similarly to derive a contradiction. If $v_f(t) = 0$, then $v_f(t') = 0$ for all $t' \geq t$, and thus $t_n < t$ for all $n \in \mathbb{Z}_+$. Again, a similar argument applies.

Conversely, suppose that (2) holds. Then $A.1'$, $A.3.1'$ and $A.4'$ are clearly implied. To check $A.2'$, let $x \in X$, $f \in \mathcal{F}_0$ and $l_1, l_2, l_3 \in \mathcal{L}_1$ be given such that $l_1 \lambda l_3 \in \mathcal{L}_1$ for some $\lambda \in (0,1)$. Assume that $(l_1, x) \succ' (f, l_2) \succ' (l_3, x)$. Note that $L_{l_1\lambda l_3}(x) = \lambda L_{l_1}(x) + (1-\lambda) L_{l_3}(x)$ for all $\lambda \in [0,1]$. Hence, there must exist $\alpha, \beta \in (0,1)$ such that $(l_1 \alpha l_3, x) \succ' (f, l_2) \succ' (l_1 \beta l_3, x)$. Assume that $(f, l_1) \succ' (l_2, x) \succ' (f, l_3)$. If either $(f, l_1) \notin \max \mathcal{M}$ or $(f, l_3) \notin \min \mathcal{M}$, then $v_f(\mathbb{R}) \neq \{0,1\}$, and thus $v_f$ is continuous. Hence, $v_f(\mathbb{R})$ is connected, and then there must exist $T \in \mathbb{R}$ such that $v_f(E_{l_1} u) > v_f(T) > V(l_2, x) > v_f(E_{l_3} u)$. Since $v_f$ is non-increasing, then $T \in (E_{l_1}u, E_{l_3}u)$. Therefore, we can find $\alpha \in (0,1)$ such that $E_{l_1\alpha l_3}u$, which implies that $(f, l_1\alpha l_3) \succ' (l_2, x)$. Next, consider a non-increasing sequence $\{D(f, l_1\frac{1}{n}l_3)\}_{n=1}^{\infty}$ of elements in $\mathbb{D}$. Note that $\cap_{n=1}^{\infty} D(f, l_1\frac{1}{n}l_3) = D(f, l_3)$. Since $c : \mathbb{D} \to \mathbb{R}$ is continuous, then $c(D(f, l_3)) = \lim_{n\to\infty} c(D(f, l_1\frac{1}{n}l_3))$. Moreover, $v_f(E_{l_3}u) = c(D(f, l_3))$, and $c(D(f, l_1\frac{1}{n}l_3))$ weakly decreases in $n$, so there must exist $N \in \mathbb{Z}_+$ such that $V(l_2, x) > c(D(f, l_1\frac{1}{N}l_3))$. Hence, $(l_2, x) \succ' (f, l_1\frac{1}{N}l_3)$.

□

*Proof of Lemma 10.* Suppose (1) holds. Since $A.3.2'$ implies $A.3.1'$, then the representation for $\succsim'$, the uniqueness of $c$ and the continuity of $v_f$ follow from Lemma 9. By $A.3.2'(3)$, for each $f \in \mathcal{F}_a$, $v_f$ is strictly decreasing on $v_f^{-1}((0,1))$. We would like to show the rest of (2). For any $f \in \mathcal{F}_0$, let $\overline{x}_f, \underline{x}_f \in X$ be such that $l_{\overline{x}_f} \succsim l_x \succsim l_{\underline{x}_f}$ for all $x \in f(S)$. Let $\lambda_f = \sup\{\lambda \in [0,1] \mid (f, l_{\overline{x}_f}\lambda l_{\underline{x}_f}) \in \max \mathcal{M}\}$. Clearly, $\lambda_f$ is well-defined since $l_{\underline{x}_f} \in \max \mathcal{M}$. Let $l_f = l_{\overline{x}_f}\lambda_f l_{\underline{x}_f}$. If $\lambda_f = 0$, then $(f, l_f) \in \max \mathcal{M}$. If $\lambda_f > 0$, then there is a non-decreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of real numbers in $[0,1]$ such that $(f, l_{\lambda_n}) \in \max \mathcal{M}$ for each $n \in \mathbb{Z}_+$, and $\lim_{n\to\infty} \lambda_n = \lambda_f$. Thus $\{D(f, l_{\lambda_n})\}_{n=1}^{\infty}$ is a non-increasing sequence of sets in $\mathbb{D}$ such that $D(f, l_f) = \cap_{n=1}^{\infty} D(f, l_{\overline{x}_f}\lambda_n l_{\underline{x}_f})$. By the upper continuity of $c$, $(f, l_f) \in \max \mathcal{M}$. Let $C = \cap_{f\in\mathcal{F}_0} D(f, l_f)$. Clearly, $C$ is closed and convex since it is the intersection of a family of closed and convex sets.

We introduce some notations which will be useful in the following proof. For any $(f, l) \in \mathcal{F}_0 \times \mathcal{L}_1$, let $\overline{H}(f, l) = \{r \in \mathbb{R}^{|S|} \mid u(f) \cdot r \geq E_l u\}$, $\underline{H}(f, l) = \{r \in \mathbb{R}^{|S|} \mid u(f) \cdot r \leq E_l u\}$, $\overline{H}^{\circ}(f, l) = \{r \in \mathbb{R}^{|S|} \mid u(f) \cdot r > E_l u\}$ and $\underline{H}^{\circ}(f, l) = \{r \in \mathbb{R}^{|S|} \mid u(f) \cdot r < E_l u\}$.

In the following, we show that for all $f \in \mathcal{F}_0$, $E_{l_f}u = \min_{p\in C} E_p u(f)$, and as a result $C$ is non-empty. Fix $f \in \mathcal{F}_0$. Since $E_{l_f}u = \min_{p\in D(f,l_f)} E_p u(f)$ and $C \subseteq D(f, l_f)$, then $E_{l_f}u \leq \min_{p\in C} E_p u(f)$. Suppose the contrary that $E_{l_f}u < \min_{p\in C} E_p u(f)$. Hence, there exists $l' \in \mathcal{L}_1$ such that $E_{l_f}u < E_{l'}u < \min_{p\in C} E_p u(f)$. We can assume without loss of generality that $E_{l'}u = 0$.

(Otherwise, consider $\bar{f} \in \mathcal{F}_0$ such that $u(\bar{f}(s)) = u(f(s)) - E_{l'}u$ for all $s \in S$. It is easy to check that $\lambda_f = \lambda_{\bar{f}}$, so $E_{l_f}u < E_{l'}u < \min_{p \in C} E_p u(f)$ if and only if $E_{l_{\bar{f}}}u < 0 < \min_{p \in C} E_p u(\bar{f})$.) Since $E_{l'}u < \min_{p \in C} E_p u(f)$, then either $C = \emptyset$, or for all $p \in C$, $E_p u(f) > E_{l'}u$ and thus $E_p u(-f) < E_{-l'}u$. In both cases, $D(-f, -l') \cap C = \emptyset$. In other words, for each $p \in D(-f, -l')$, there exists $g \in \mathcal{F}_0$ such that $p \notin D(g, l_g)$, or $p \in \underline{H}^\circ(g, l)$ for some $l \in \mathcal{L}_1$ satisfying $l_g \succ l$. Since $D(-f, -l')$ is compact, there exist $g_1, ..., g_N \in \mathcal{F}_0$ and $l_1, ..., l_N \in \mathcal{L}_1$ such that $l_{g_n} \succ l_n$ for all $n = 1, ..., N$, and $D(-f, -l') \subseteq \cup_{n=1}^N \underline{H}^\circ(g_n, l_n)$. Let $x_0, x_1 \in X$ be such that $u(x_0) = 0$ and $u(x_1) = 1$. Let $S = \{s_1, ..., s_{|S|}\}$, and let $h_m \in \mathcal{F}_0$ be such that $h_m(s_m) = x_1$ and $h_m(s) = x_0$ when $s \neq s_m$, $m = 1, ..., |S|$. Note that $\Delta(S) \subseteq \cap_{m=1}^{|S|} \overline{H}(h_m, l_{x_0})$, and thus $D(-f, -l') \subseteq [\cup_{n=1}^N \underline{H}^\circ(g_n, l_n)] \cap [\cap_{m=1}^{|S|} \overline{H}(h_m, l_{x_0})]$. Again without loss of generality, we assume that $E_{l_n}u = 0$, $n = 1, ..., N$, otherwise we can change $g_1, ..., g_N$ so that this holds while the relations above remain the same. Next we quote Farkas' lemma.

**Lemma 12** (Farkas' lemma)**.** *For any $i \times j$ matrix $B$ and $i$-dimensional vector $b$, $i, j \in \mathbb{Z}_+$, exactly one of the following two statements is true.*

*(i) There exists $q \in \mathbb{R}^j$ such that $Bq = b$ and $b \geq \mathbf{0}$.*

*(ii) There exists $r \in \mathbb{R}^i$ such that $B^T r \geq \mathbf{0}$ and $b^T r < 0$.*

We consider $u(f)$ as a vector in $\mathbb{R}^{|S|}$, i.e., $u(f) = (u(f(s_1)), ..., u(f(s_{|S|})))^T$. Let $b = u(f)$, and similarly let $B = [u(g_1), ..., u(g_N), u(h_1), ..., u(h_{|S|})]$ be a $|S| \times (N + |S|)$ matrix. Suppose that $r \in \mathbb{R}^S$ and $B^T r \geq \mathbf{0}$. That is, $u(g_n)^T r \geq 0$ for all $n = 1, ..., N$, and $u(h_m)^T r \geq 0$, i.e. $r_m \geq 0$, for all $m = 1, ..., |S|$. Thus, $r \in [\cap_{n=1}^N \overline{H}(g_n, l_n)] \cap [\cap_{m=1}^{|S|} \overline{H}(h_m, l_{x_0})]$. If $r = \mathbf{0}$, then $b^T r = 0$. If $r \neq \mathbf{0}$, then $\sum_{m=1}^{|S|} r_m > 0$. Since $u(g_n)^T \frac{r}{\sum_{m=1}^{|S|} r_m} \geq 0$ for each $n = 1, ..., N$, then $\frac{r}{\sum_{m=1}^{|S|} r_m} \notin D(-f, -l')$. That is, $u(-f)^T \frac{r}{\sum_{m=1}^{|S|} r_m} < E_{-l'}u$, and thus $u(f)^T \frac{r}{\sum_{m=1}^{|S|} r_m} > E_{l'}u = 0$. Therefore, (ii) does not hold, and there exists $q \geq \mathbf{0}$ in $\mathbb{R}^j$ such that $Bq = b$. If $b = \mathbf{0}$, then $E_{l_f}u = 0$ which contradicts with $E_{l_f}u < E_{l'}u = 0$. Hence, $b \neq \mathbf{0}$, and then $q \neq \mathbf{0}$. Thus, $B \frac{q}{\sum_{n=1}^{N+|S|} q_n} = \frac{b}{\sum_{n=1}^{N+|S|} q_n}$ where $\sum_{n=1}^{N+|S|} q_n > 0$.

For all $n = 1, ..., N$, fix $-g \in \mathcal{F}_0$, $-l_n, -l_{g_n} \in \mathcal{L}_1$ such that $u(-g) = -u(g)$, $E_{-l_n}u = -E_{l_n}u$ and $E_{-l_{g_n}}u = -E_{l_{g_n}}u$. Since $(g_n, l_{g_n}) \in \max \mathcal{M}$ and $-l_n \succ -l_{g_n}$, then $(-g_n, -l_n) \in \min \mathcal{M}$ by $A.5'$, $n = 1, ..., N$. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of elements in $X$ such that $u(x_k) < 0$ for all $k \in \mathbb{Z}_+$ and $\lim_{n\to\infty} u(x_n) = 0$. For all $m = 1, ..., |S|$ and $k \in \mathbb{Z}_+$, fix $-h_m \in \mathcal{F}_0$, $-l_{x_k} \in \mathcal{L}_1$ such that $u(-h_m) = -u(h_m)$ and $E_{-l_{x_k}}u = -E_{l_{x_k}}u$. Since $(h_m, l_{x_0}) \in \max \mathcal{M}$ and $-l_{x_k} \succ l_{x_0}$, then $(-h_m, -l_{x_k}) \in \min \mathcal{M}$ by $A.5'$. By $A.6'$, $(\sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n}(-g_n) + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n}(-h_m), \sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n}(-l_n) + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n}(-l_{x_k})) \in \min \mathcal{M}$, $k \in \mathbb{Z}_+$. By the upper continuity of $c$, $(\sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n} g_n + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n} h_m, \sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n} l_n + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n} l_{x_k}) \in \max \mathcal{M}$, $k \in \mathbb{Z}_+$. Let $f' \in \mathcal{F}_0$ such that $u(f') = \frac{u(f)}{\sum_{n=1}^{N+|S|} q_n}$. Since $B\frac{q}{\sum_{n=1}^{N+|S|} q_n} = \frac{b}{\sum_{n=1}^{N+|S|} q_n}$ and $b = u(f)$, then $(f', l') \in \max \mathcal{M}$. Since $E_p u(f') \geq E_{l'}u$ if and only if $E_p u(f) \geq E_{l'}u \sum_{n=1}^{N+|S|} q_n = E_{l'}u$, then $(f, l') \in \max \mathcal{M}$. Since $l' \succ l_f$, this contradicts the choice of $l_f$.

Next we show that $c$ is compatible with the set $C$. Suppose that $D_1 = D(f_1, l_1)$, $D_2 = D(f_2, l_2)$ and $D_1 \cap C \supseteq D_2 \cap C$. If $D_2 \cap C = \emptyset$, then $E_{l_2}u > \max_{p\in C} E_p u(f_2)$. Fix $-f_2 \in \mathcal{F}_0$, $-l_2 \in \mathcal{L}_1$ such that $u(-f_2) = -u(f_2)$ and $E_{-l_2}u = -E_{l_2}u$. Thus, $E_{-l_2}u < \min_{p\in C} E_p u(-f_2) = E_{l_{-f_2}}u$. Since $(-f_2, l_{-f_2}) \in \max \mathcal{M}$ and $l_{-f_2} \succ -l_2$, then $l_2 \succ -l_{-f_2}$ and thus by $A.5'$, $(f_2, l_2) \in \min \mathcal{M}$. Hence, $c(D_1) \geq 0 = c(D_2)$. If $D_1 \cap C = C$, then $C \subseteq D_1$, that is, $E_{l_1}u \leq \min_{p\in C} E_p u(f_1) = E_{l_{f_1}}u$. Since $(f_1, l_{f_1}) \in \max \mathcal{M}$ and $l_{f_1} \succsim l_1$, then $(f_1, l_1) \in \max \mathcal{M}$ by $A.3.2'$. Hence, $c(D_1) = 1 \geq c(D_2)$.

Suppose that $C \supsetneq D_1 \cap C \supseteq D_2 \cap C \supsetneq \emptyset$. Let $A = \{r \in \mathbb{R}^{|S|} \mid u(f_1)^T r < E_{l_1}u$ and $u(f_2)^T r \geq E_{l_2}u\}$. If $A \cap \Delta(S) = \emptyset$, then for each $p \in \Delta(S)$, $(f_2, p) \succsim l_2$ implies that $(f_1, p) \succsim l_1$, i.e., $D_1 \supseteq D_2$. Thus, $c(D_1) \geq c(D_2)$. Suppose that $A \cap \Delta(S) \neq \emptyset$. Thus, $A \neq \emptyset$, and it is easy to see that the interior of $A$ is non-empty. Since $D_1 \cap C \supseteq D_2 \cap C$, then $A \cap C = \emptyset$. Therefore, by a basic separation theorem (Dunford and Schwartz (1966), V.1.12), there exists a non-zero linear functional $I$ on $\mathbb{R}^{|S|}$ and a real number $\lambda$ such that $I(r) \geq \lambda \geq I(r')$ for all $r \in C$ and $r' \in A$. Let $e_n$ be a vector in $\mathbb{R}^{|S|}$ that takes 1 in the $n$-th coordinate and 0 in the other coordinates, $n = 1, ..., |S|$. Pick $f_3 \in \mathcal{F}_0$ such that $u(f_3(s_n)) = I(e_n)$, $n = 1, ..., |S|$. Then $I(r) = u(f_3)^T r$ for each $r \in \mathbb{R}^{|S|}$. Pick $l_3 \in \mathcal{L}_1$ such that $E_{l_3}u = \lambda$. Since $I(C) \geq \lambda$, then $\min_{p \in C} E_p u(f_3) \geq E_{l_3}u$, and thus $(f_3, l_3) \in \max \mathcal{M}$.

If $(f_2, l_2) \in \max \mathcal{M}$, then $C \subseteq D_2$ by the construction of $C$. Thus $C = D_2 \cap C$ which contradicts our assumption. Hence, either $v_{f_2}(E_{l_2}u) \in (0, 1)$ or $(f_2, l_2) \in \min \mathcal{M}$. In both cases, there exists a sequence $\{l'_n\}_{n=1}^{\infty}$ of elements in $\mathcal{L}_1$ such that $l'_n \succ l_2$ for all $n \in \mathbb{Z}_+$, and $\lim_{n \to \infty} v_{f_2}(E_{l'_n}u) = v_{f_2}(E_{l_2}u)$. Fix $n \in \mathbb{Z}_+$. Fix $p \in \Delta(S)$ such that $(f_3, p) \succsim l_3$ and $(f_2, p) \succsim l'_n$, i.e., $E_p u(f_3) \geq \lambda$ and $E_p u(f_2) \geq E_{l'_n}u > E_{l_2}u$. If $p \in A$, then $p$ is an interior point of $A$, and thus $E_p u(f_3) < \lambda$ which is a contradiction. Hence, $p \notin A$, and thus $(f_1, p) \succsim l_1$. By $A.3.2'$, $(f_1, l_1) \succsim' (f_2, l'_n)$. Taking the limit, we get that $c(D_1) \geq c(D_2)$.

Let $(f, l) \in \mathcal{M}$ be given. Suppose $\max\{(f, p) \mid p \in C\} \succ l$ and $(f, l) \in \min \mathcal{M}$. Then by $A.5'$ and the upper continuity of $c$, $(-f, -l) \in \max \mathcal{M}$. Note that $E_{-l}u > \min_{p \in C} E_p u(-f) = E_{l_{-f}}u$. Thus, $-l \succ l_{-f}$ which is a contradiction to the construction of $l_{-f}$. Hence, $(f, l) \notin \min \mathcal{M}$ and $c(D(f, l)) > 0$. Suppose $l \succ \min\{(f, p) \mid p \in C\}$. Since $E_{l_f}u = \min_{p \in C} E_p u(f)$, then by the choice of $l_f$, $(f, l) \notin \max \mathcal{M}$. As a result, $c(D(f, l)) < 1$.

To check the uniqueness of $C$, suppose the contrary that there exists $C' \subseteq \Delta(S)$ satisfying the desirable properties. Suppose that $q \in C' \setminus C$. Then there exists $f \in \mathcal{F}_0$ such that $q \notin D(f, l_f)$, and thus $\min_{p \in C'} E_p u(f) \leq E_q u(f) < E_{l_f}u$. Since $c$ is compatible with $C'$, then $c(D(f, l_f)) < 1$ which is a contradiction. Suppose that $q \in C \setminus C'$. By a separation theorem (Dunford and Schwartz (1966), V.2.10), there exists a linear functional $I$ on $\mathbb{R}^{|S|}$ such that $I(q) < \min_{p \in C'} I(p)$. Pick $f \in \mathcal{F}_0$ such that $u(f(s_n)) = I(e_n)$, $n = 1, ..., |S|$. Hence, for all $p \in \Delta(S)$, $I(p) = E_p u(f)$. Since $q \in C$, then $\min_{p \in C} E_p u(f) \leq E_q u(f) < \min_{p \in C'} E_p u(f)$. Let $l \in \mathcal{L}_1$ be such that $\min_{p \in C} E_p u(f) < E_l u < \min_{p \in C'} E_p u(f)$. Since $c$ is compatible with $C$, then $c(D(f, l)) < 1$. Since $c$ is compatible with $C'$ and $D(f, l) \cap C' = C'$, then $c(D(f, l)) = 1$, which is a contradiction as desired.

Next, we show that $\mathcal{U} = \{A \subseteq S \mid p(A) = p'(A) \text{ for all } p, p' \in C\}$, which implies that $\mathcal{U}$ is a $\lambda$-system. Suppose that $A \in \mathcal{U}$. Let $x_0, x_1, x_1' \in X$ be given such that $u(x_0) = 0$, $u(x_1) = 1$ and $u(x_1') = -1$. Since $A$ is unambiguous, then $(x_1Ax_0, l_{x_1}\lambda l_{x_0}) \in \min \mathcal{M}$ when $\lambda > \lambda_{x_1Ax_0}$. Thus, $(x_1'Ax_0, l_{x_1'}\lambda l_{x_0}) \in \max \mathcal{M}$ when $\lambda > \lambda_{x_1Ax_0}$. By the upper continuity of $c$, $(x_1'Ax_0, l_{x_1'}\lambda_{x_1Ax_0}l_{x_0}) \in \max \mathcal{M}$. Hence, $\lambda_{x_1'Ax_0} \geq 1 - \lambda_{x_1Ax_0}$. Suppose $\lambda_{x_1'Ax_0} > 1 - \lambda_{x_1Ax_0}$. Then $l_{x_0}\lambda_{x_1'Ax_0}l_{x_1'} > l_{x_1'}\lambda_{x_1Ax_0}l_{x_0}$. Since $(x_1Ax_0, l_{x_1}\lambda_{x_1Ax_0}l_{x_0}) \in \max \mathcal{M}$, then by $A.5'$, $(x_1'Ax_0, l_{x_0}\lambda_{x_1'Ax_0}l_{x_1'}) \in \min \mathcal{M}$, which contradicts the choice of $\lambda_{x_1'Ax_0}$. Hence, $\lambda_{x_1'Ax_0} = 1 - \lambda_{x_1Ax_0}$. For any $p \in C$, $p \in D(x_1Ax_0, l_{x_1Ax_0}) \cap D(x_1'Ax_0, l_{x_1'Ax_0})$. That is, $p(A) \geq \lambda_{x_1Ax_0}$ and $-p(A) \geq -(1 - \lambda_{x_1'Ax_0}) = -\lambda_{x_1Ax_0}$. Hence, $p(A) = \lambda_{x_1Ax_0}$ for all $p \in C$.

On the other hand, suppose that $A \subseteq S$ and $p(A) = p'(A)$ for all $p, p' \in C$. Let $x, y \in X$ be given. Note that $\max_{p \in C} E_p u(xAy) = \min_{p \in C} E_p u(xAy) = E_{l_{xAy}} u$. Fix $l \in \mathcal{L}_1$. If $l_{xAy} \gtrsim l$, then $(xAy, l) \in \max \mathcal{M}$. If $l > l_{xAy}$, then $D(xAy, l) \cap C = \emptyset$, and thus $c(D(xAy, l)) = 0$, i.e., $(xAy, l) \in \min \mathcal{M}$. Hence, $A \in \mathcal{U}$.

Conversely, suppose that (2) holds. By Lemma 9, $A.1'$, $A.2'$, $A.3.2'(1)$ and $A.4'$ holds. To see $A.3.2'(2)$, note that $(f_3, l_3) \in \max \mathcal{M}$ implies that $\min\{(f_3, p) \mid p \in C\} \gtrsim l_3$, and thus $C \subseteq D(f_3, l_3)$. Hence, if $(f_2, p) \gtrsim l_2$ implies $(f_1, p) \gtrsim l_1$ either for each $p \in \Delta(S)$ or for each $p \in D(f_3, l_3)$ where $(f_3, l_3) \in \max \mathcal{M}$, then $D(f_1, l_1) \cap C \supseteq D(f_2, l_2) \cap C$, so that $(f_1, l_1) \gtrsim' (f_2, l_2)$. $A.3.2'(3)$ follows from that $v_f$ is strictly decreasing on $v_f^{-1}((0, 1))$ if $f \in \mathcal{F}_a$.

Finally, we check $A.5'$. Suppose $(f, l) \in \max \mathcal{M}$. Then $C \subseteq D(f, l)$. If $l' > -l$, then $D(-f, l') \cap D(f, l) = \emptyset$, and thus $D(-f, l') \cap C = \emptyset$. Hence, $D(-f, l') \in \min \mathcal{M}$. Suppose $(f, l) \in \min \mathcal{M}$. Then $l \gtrsim \max\{(f, p) \mid p \in C\}$, and thus $\min_{p \in C} E_p u(-f) \geq E_{-l} u$. It follows that $C \subseteq D(-f, -l) \subseteq D(-f, l')$ if $-l > l'$. Hence, $(-f, l') \in \max \mathcal{M}$. Suppose that $(f_1, l_1), (f_2, l_2) \in \min \mathcal{M}$ and $\lambda \in (0, 1)$. Then $E_{l_1} \geq \max_{p \in C} E_p u(f_1)$ and $E_{l_2} \geq \max_{p \in C} E_p u(f_2)$. For any $p \in C$, $E_p u(f_1 \lambda f_2) = \lambda E_p u(f_1) + (1 - \lambda) E_p u(f_2) \leq \lambda E_{l_1} u + (1 - \lambda) E_{l_2} u = E_{l_1 \lambda l_2} u$. If

$\max_{p\in C} E_p u(f_1\lambda f_2) < E_{l_1\lambda l_2}u$, then $D(f_1\lambda f_2, l_1\lambda l_2)\cap C = \emptyset$, and thus $(f_1\lambda f_2, l_1\lambda l_2) \in \min \mathcal{M}$. If $\max_{p\in C} E_p u(f_1\lambda f_2) = E_{l_1\lambda l_2}u$, then $E_{l_n}u = \max_{p\in C} E_p u(f_n)$ for each $n = 1, 2$. Hence, $\max_{p\in C} E_p u(f_i) > \min_{p\in C} E_p u(f_n)$ for each $n = 1, 2$, otherwise $(f_1, l_1), (f_2, l_2) \in \max \mathcal{M}$. Let $q \in \arg\min_{p\in C} E_p u(f_1)$. Then $\min_{p\in C} E_p u(f_1\lambda f_2) \leq E_q u(f_1\lambda f_2) < \lambda \max_{p\in C} E_p u(f_1) + (1-\lambda)\max_{p\in C} E_p u(f_2) = E_{l_1\lambda l_2}u$. Since $\min_{p\in C} E_p u(f_1\lambda f_2) < E_{l_1\lambda l_2}u = \max_{p\in C} E_p u(f_1\lambda f_2)$, then $f_1\lambda f_2 \in \mathcal{F}_a$. By the continuity of $v_{f_1\lambda f_2}$, $v_{f_1\lambda f_2}(E_{l_1\lambda l_2}u) = 0$, i.e., $(f_1\lambda f_2, l_1\lambda l_2) \in \min \mathcal{M}$.

□

*Proof of Theorem 1.* We only check the preference representation result. Suppose (1) holds. We gradually prove the results by establishing the following facts.

**F.1.** For all $f \in \mathcal{F}_0$ and $l \in \mathcal{L}_1$ such that $l_{f(s)} \sim l$ for each $s \in S$, then $f \sim l$.

Pick $x \in X$ such that $l_x \sim l$. By *A*.2.2, $f_x \sim l_x$ and thus $f_x \sim l$. Since $(f, l) \sim' (f_x, l)$ for all $l \in \mathcal{L}_1$, then by *A*.6, $f \sim f_x \sim l$.

**F.2.** For all $f, g \in \mathcal{F}_0$, if $l_{f(s)} \gtrsim l_{g(s)}$ for all $s \in S$, then $f \gtrsim g$.

For any $l \in \mathcal{L}_1$ and $p \in \Delta(S)$, $(g, p) \gtrsim l$ implies that $(f, p) \gtrsim l$, and thus $(f, l) \gtrsim '(g, l)$ by *A*.3.2′(2). Hence, $f \gtrsim g$ by *A*.6.

**F.3.** For any $f \in \mathcal{F}_0$, there exists $l \in \mathcal{L}_1$ such that $f \sim l$. Moreover, if $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, then $f \sim \underline{l}_f$.

Let $f_0, f_1 \in \mathcal{F}_0$ be such that for all $s \in S$, $l_{f_0(s)} \sim \underline{l}_f$ and $l_{f_1(s)} \sim \bar{l}_f$. By *A*.6, $f_1 \gtrsim f \gtrsim f_0$. If $f \sim f_0$ or $f \sim f_1$, then $f \sim \underline{l}_f$ or $f \sim \bar{l}_f$ by F.1. Suppose that $f_1 \succ f \succ f_0$. By *A*.3.2, there exist $l', l'' \in \mathcal{L}_1$ such that $f_1 \succ l' \succ f \succ l'' \succ f_0$. Thus, $\bar{l}_f \succ l' \succ f \succ l'' \succ \underline{l}_f$. One can check that there exists $\gamma \in (0, 1)$ such that $\bar{l}_f \lambda \underline{l}_f \succ f$ if $\lambda \in (\gamma, 1]$ and $f \succ \bar{l}_f \lambda \underline{l}_f$ if $\lambda \in [0, \gamma)$. If $f \succ \bar{l}_f \gamma \underline{l}_f$, then by F.1, $f \succ f_1 \gamma f_0$. By $A.3.2$, there exists $l''' \in \mathcal{L}_1$ such that $f \succ l''' \succ f_1 \gamma f_0$. Thus, $\bar{l}_f \succ f \succ l''' \succ \bar{l}_f \gamma \underline{l}_f$, and then $l''' \sim \bar{l}_f \lambda \underline{l}_f$ for some $\lambda \in (\gamma, 1)$ which is a contradiction to the construction of $\gamma$. Similarly, it is not true that $\bar{l}_f \gamma \underline{l}_f \succ f$. Hence, $f \sim \bar{l}_f \gamma \underline{l}_f$, as desired.

If $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, then $\underline{l}_f \sim \bar{l}_f$ and thus $f_1 \sim f_0$. Hence, $f \sim f_0 \sim \underline{l}_f$.

Define $I : \mathbb{R}^S \to \mathbb{R}$ by $I(r) = E_l u$ if where $r \in \mathbb{R}^S$, $f \in \mathcal{F}_0$ and $l \in \mathcal{L}_1$ satisfy $r = u(f)$ and $f \sim l$. By $F.2$ and $F.3$, $I$ is well-defined. Recall that $e \in \mathbb{R}^S$ is the unit vector which assigns 1 to every coordinate. It is easy to see that (1) $I(u(f)) \geq I(u(g))$ if and only if $f \succsim g$, (2) $I(te) = t$ for all $t \in \mathbb{R}$, and (3) $I(r) \geq I(r')$ for all $r \geq r'$ in $\mathbb{R}^S$. For any $f \in \mathcal{F}_0$ and $t \in \mathbb{R}$, let $f_t$ denote an act in $\mathcal{F}_0$ such that $u(f_t) = u(f) + te$. Note that for all $f \in \mathcal{F}_0$, $t, k \in \mathbb{R}$, $v_f(k) = v_{f_t}(k + t)$, and thus, $E_{\underline{l}_{f_t}} u = E_{\underline{l}} u + t$ and $E_{\bar{l}_{f_t}} u = E_{\bar{l}} u + t$.

**F.4.** There exist $k, k' > 0$ such that $I(u(f) + te) - I(u(f)) \in [k't, kt]$ for all $t > 0$ and $f \in \mathcal{F}_a$.

Suppose $x_0 \in X$ and $u(x_0) = 0$. Let $f \in \mathcal{F}_a$ and $t > 0$ be given. Let $\alpha, \beta \in (0, 1)$ be given as in $A.4.2$. Suppose $x_f \in X$ and $f \sim l_{x_f}$. Note that $x_f$ always exists. Then $I(u(f)) = I(u(x_f)e) \to I(\alpha\frac{u(f)}{\alpha} + (1 - \alpha)\mathbf{0}) = I(\beta\frac{u(x_f)e}{\beta} + (1 - \beta)\mathbf{0}) \to I(u(\alpha f' + (1 - \alpha)f_{x_0})) = I(u(\beta g' + (1 - \beta)f_{x_0}))$ where $f', g' \in \mathcal{F}_0$ satisfy $u(f') = \frac{u(f)}{\alpha}$ and $u(g') = \frac{u(x_f)e}{\beta}$. Hence, $\alpha f' + (1 - \alpha)f_{x_0} \sim \beta g' + (1 - \beta)f_{x_0}$. Let $y \in X$ and $u(y) = \frac{t}{1-\alpha}$. Since $g' \in \mathcal{F}_0 \setminus \mathcal{F}_a$ and $y \succ x_0$, then $\beta\frac{g'}{\beta} + (1 - \beta)f_y \succsim \alpha f' + (1 - \alpha)f_y$. Thus, $\beta\frac{u(x_f)}{\beta} + (1 - \beta)0 + (1 - \beta)\frac{t}{1-\alpha} \geq I(u(\alpha f' + (1 - \alpha)f_y)) \to I(u(\beta g' + (1 - \beta)f_{x_0})) + \frac{1-\beta}{1-\alpha}t \geq I(u(f) + te) \to I(u(\alpha f' + (1 - \alpha)f_{x_0})) + \frac{1-\beta}{1-\alpha}t \geq I(u(f) + te) \to I(u(f) + te) - I(u(f)) \leq \frac{1-\beta}{1-\alpha}t$. Similarly, $I(u(f) + te) - I(u(f)) \geq \frac{1-\gamma}{1-\alpha}t$. Hence, $k' = \frac{1-\gamma}{1-\alpha}$ and $k = \frac{1-\beta}{1-\alpha}$.

**F.5.** For any $f, g \in \mathcal{F}_0$, if $f(s) \succ g(s)$ for all $s \in S$, then $f \succ g$.

Let $t = \min_{s \in S}[u(f(s)) - u(g(s))]$. Note that $t > 0$ and $u(g) \geq u(f) + te$. By $F.2$, it suffices to show that $I(u(f) + te) > I(u(f))$, which follows from $F.3$ and $F.4$.

**F.6.** For all $x \in X$ and $f \in \mathcal{F}_a$, there exists $t \in \mathbb{R}$ such that $f_t \sim f_x$.

Fix $x \in X$ and $f \in \mathcal{F}_a$. Pick $t_0, t_1 \in \mathbb{R}$ such that $\min_{s \in S} u(f_{t_1}(s)) \geq u(x) \geq \max_{s \in S} u(f_{t_0})$. By $F.2$, $f_{t_1} \succsim f_x \succsim f_{t_0}$. If $f_{t_1} \sim f_x$ or $f_{t_0} \sim f_x$, then by $A.2.2$, we are done. Suppose that $f_{t_1} \succ f_x \succ f_{t_0}$. Thus, $t_1 > t_0$ by $F.2$. Fix $f_{t_1} \lambda f_{t_0} \in \mathcal{F}$ where $\lambda \in [0, 1]$. Note that $u(f_{t_1} \lambda f_{t_0}) = u(f_{t_0}) + \lambda(t_1 - t_0)e$ and $f_{t_0} \in \mathcal{F}_a$. By $F.4$, $I(u(f_{t_1} \lambda f_{t_0})) \leq I(u(f_{t_0})) + k\lambda(t_1 - t_0)$ for some $k \in \mathbb{R}$. Hence, there exists $\lambda \in (0, 1)$ such that $f_x \succ f_{t_1} \lambda f_{t_0}$. Similarly, there exists $\lambda' \in (0, 1)$ such that $f_{t_1} \lambda f_{t_0} \succ f_x$. Hence, we can find $\gamma \in (0, 1)$ such that $f_x \succ f_{t_1} \lambda f_{t_0}$ for all $\lambda \in [0, \gamma)$ and $f_{t_1} \lambda f_{t_0} \succ f_x$ for all $\lambda \in (\gamma, 1]$. Suppose that $f_x \succ f_{t_1} \gamma f_{t_0}$. When $\lambda \in (\gamma, 1]$, $u(f_{t_1} \lambda f_{t_0}) = u(f_{t_1} \gamma f_{t_0}) + (\lambda - \gamma)(t_1 - t_0)$, and then $I(u(f_{t_1} \lambda f_{t_0})) \leq I(u(f_{t_1} \gamma f_{t_0})) + k(\lambda - \gamma)(t_1 - t_0)$. Hence, there exists $\lambda \in (\gamma, 1]$ such that $f_x \succ f_{t_1} \lambda f_{t_0} \succ f_{t_1} \gamma f_{t_0}$, which is a contradiction to the construction of $\gamma$. Similarly, it is not true that $f_{t_1} \gamma f_{t_0} \succ f_x$. Hence, $f_{\gamma t_1 + (1-\gamma) t_0} \sim f_{t_1} \gamma f_{t_0} \sim f_x$.

Suppose that $\mathcal{F}_a \neq \emptyset$. For each $x \in X$, let $F_x = \{f \in \mathcal{F}_a \mid f \sim f_x\}$ and $U_x = \cap_{f \in F_x} \{r \in \mathbb{R} \times [0, 1] \mid r_1 \geq E_{l_f} u, r_2 \geq v_f(r_1)\}$. Note that $F_x \neq \emptyset$. Moreover, $U_x$ is closed in $\mathbb{R} \times [0, 1]$ since $v_f$ is continuous when $f \in \mathcal{F}_a$.

**F.7.** Suppose that $\mathcal{F}_a \neq \emptyset$. For any $x, y \in X$, if $f_y \succ f_x$, then $U_x \supseteq U_y$.

Suppose that $r \in U_y$ and $f \in F_x$. Then there exists $t \in \mathbb{R}$ such that $f_t \sim f_y$. Hence, $f_t \in F_y$. Since $r \in U_y$ and $f_t \sim f_y \succ f_x \sim f$, then $r_1 \geq E_{\underline{l}_{f_t}} u \geq E_{\underline{l}_f} u$ and $r_2 \geq v_{f_t}(r_1) = v_f(r_1 - t) \geq v_f(r_1)$. Hence, $r \in U_x$.

**F.8.** Suppose that $\mathcal{F}_a \neq \emptyset$. For any $x \in X$, $(u(x), 1) \in U_x$, and thus $U_x \neq \emptyset$.

Fix $f \in F_x$. It suffices to show that $u(x) \geq E_{\underline{l}_f} u$. Suppose the contrary that $u(x) < E_{\underline{l}_f} u$. There exists $t < 0$ such that $u(x) < E_{\underline{l}_{f_t}} u$. By $A.6$, $f_t \succsim f_x$. Hence, $f \succ f_t \succsim f_x$ which is a contradiction to $f \sim f_x$.

**F.9.** Suppose that $\mathcal{F}_a \neq \emptyset$. Let $x, y \in X$ be given. If $f_y \succ f_x$, then $(u(x), k) \notin U_y$ for all $k \in [0, 1]$.

Since $f_y \succ f_x$, then there must exist $f \in \mathcal{F}_a$ such that $u(y) > E_{\bar{l}_f} u > E_{\underline{l}_f} u > u(x)$. Suppose that $f \sim f_z$. Thus, $(u(x), k) \notin U_z$ for all $k \in [0, 1]$, and $f_y \succ f_z$. By F.7, $(u(x), k) \notin U_y$ for all $k \in [0, 1]$.

**F.10.** Let $x, y \in X$ be given so that $f_y \succ f_x$. Suppose that $f \in F_x$ and $(k, v_f(k)) \in U_x$ where $k \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]$. Then $(k, v_f(k)) \notin U_y$.

Since $f_y \succ f_x \sim f$, then there exists $t > 0$ such that $f_t \sim f_y$. If $k = E_{\underline{l}_f} u$, then $k < E_{\underline{l}_{f_t}} u$, and thus $(k, v_f(k)) \notin U_y$. If $E_{\underline{l}_f} u < k \leq E_{\bar{l}_f} u$, then $v_{f_t}(k) = v_f(k - t) > v_f(k)$ by the strict-increasing property of $v_f$. Hence, $(k, v_f(k)) \notin U_y$.

**F.11.** Suppose that $\mathcal{F}_a \neq \emptyset$ and $(r_1, r_2) \in U_x$ for some $x \in X$. If $r_1' \geq r_1$, then $(r_1', r_2) \in U_x$.

If there exists $r_1' < r_1$ such that $(r_1', r_2) \in U_x$, then there exists $y \in X$ such that $f_y \succ f_x$ and $(r_1, r_2) \in U_y$.

If $r_1' \geq r_1$, then for any $f \in F_x$, $r_1' \geq r_1 \geq E_{\underline{l}_f} u$ and $r_2 \geq v_f(r_1) \geq v_f(r_1')$. Hence, $(r_1', r_2) \in U_x$.

Suppose that there exists $r_1' < r_1$ such that $(r_1', r_2) \in U_x$. Let $R = r_1 - r_1'$. Let $k' > 0$ be given as in $F.4$. By F.9, $r_1 > r_1' \geq u(x)$. Suppose the contrary that for all $y \in X$ such that $f_y \succ f_x$, $(r_1, r_2) \notin U_y$. Pick $y^* \in X$ such that $u(y^*) \in (u(x), r_1]$ and $u(y^*) - u(x) < k'R$. There exists $f \in F_{y^*}$ such that $r_2 < v_f(r_1)$, since $r_1 \geq u(y^*) \geq E_{\underline{l}_f} u$. By $F.4$, $I(u(f)) - I(u(f_{-R})) \geq k'R$. Then $I(u(f_{-R})) \leq I(u(f)) - k'R = u(y^*) - k'R < u(x)$. Hence, by $F.5$ and $F.6$, there exists $\triangle R > 0$ such that $f_{-R+\triangle R} \sim f_x$. Thus, $v_{f_{-R+\triangle R}}(r_1') \geq v_{f_{-R+\triangle R}}(r_1' + \triangle R) = v_f(r_1) > r_2$, which is a contradiction to $(r_1', r_2) \in U_x$.

**F.12.** Suppose that $\mathcal{F}_a \neq \emptyset$ and $(r_1, r_2) \notin U_x$ for all $x \in X$. Then for any $r_1' \in \mathbb{R}$, $(r_1', r_2) \notin U_x$ for all $x \in X$.

If $r_1' \leq r_1$, then by $F.11$, $(r_1', r_2) \notin U_x$ for all $x \in X$. Suppose that $r_1' > r_1$. Let $R = r_1' - r_1$. Let $k > 0$ be given as in $F.4$. Suppose the contrary that $r_1' > r_1$ and $(r_1', r_2) \in U_y$ for some $y \in X$. Pick $x^* \in X$ such that $u(x^*) < \min\{r_1, u(y) - kR\}$. Since $(r_1, r_2) \notin U_{x^*}$ and $u(x^*) < r_1$, then there exists $f \in \mathcal{F}_{x^*}$ such that $v_f(r_1) > r_2$. Note that $I(u(f_R)) \leq I(u(f)) + kR$. Thus, $I(u(f_R)) \leq u(x^*) + kR < u(y)$. There exists $\triangle R > 0$ such that $f_{R+\triangle R} \sim f_y$. Hence, $v_{f_{R+\triangle R}}(r_1') = v_{(f_R}(r_1' - \triangle R) \geq v_{f_R}(r_1') = v_f(r_1) > r_2$, which is a contradiction to $(r_1', r_2) \in U_y$, as desired.

Define $w : \mathbb{R} \times [0, 1] \to [-\infty, \infty)$ by $w(r_1, r_2) = r_1$ if $\mathcal{F}_a = \emptyset$ and

$w(r_1, r_2) = \sup\{u(x) \mid x \in X \text{ and } (r_1, r_2) \in U_x\}$ if $\mathcal{F}_a \neq \emptyset$.

**F.13.** The function $w$ is well-define and normalized.

The result is obvious when $\mathcal{F}_a = \emptyset$. Suppose that $\mathcal{F}_a \neq \emptyset$. To see $w$ is well-defined, just note that by *F*.9, $\{u(x) \mid x \in X \text{ and } (r_1, r_2) \in U_x\}$ is bounded above for any $(r_1, r_2) \in \mathbb{R} \times [0, 1]$. The fact that $w$ is normalized follows from *F*.8 and *F*.9.

**F.14.** Suppose that $\mathcal{F}_a \neq \emptyset$ and $w(r_1, r_2) = u(x)$ for some $x \in X$. Then $(r_1, r_2) \in U_x$.

Suppose the contrary that $(r_1, r_2) \notin U_x$. Hence, there exists $f \in F_x$ such that either $r_1 < E_{\underline{l}_f} u$ or $r_2 < v_f(r_1)$. If $r_1 < E_{\underline{l}_f} u$, then pick $t \in (0, E_{\underline{l}_f} u - r_1)$ so that $r_1 < E_{\underline{l}_{f_{-t}}} u$. Thus, $(r_1, r_2) \notin U_y$ if $f_y \gtrsim f_{-t}$. Hence, $w(r_1, r_2) \leq I(u(f_{-t})) < I(u(f)) = u(x)$, which is a contradiction. If $r_2 < v_f(r_1)$, then pick $t \in (0, \min\{v_f^{-1}(r_2)\})$ so that $r_2 < v_{f_{-t}}(r_1)$ and thus $(r_1, r_2) \notin U_y$ when $f_y \gtrsim f_{-t}$. This leads to the same contradiction.

**F.15.** The function $w$ is upper semicontinuous.

The case when $\mathcal{F}_a = \emptyset$ is clear. Suppose that $\mathcal{F}_a \neq \emptyset$. Note that $U_x$ is a closed set for any $x \in X$. Thus, it suffices to show that for each $t \in \mathbb{R}$, $w^{-1}([t, \infty)) = U_x$ if $u(x) = t$. Let $t \in \mathbb{R}$ and $x \in X$ be given such that $u(x) = t$. For all $(r_1, r_2) \in U_x$, $w(r_1, r_2) \geq u(x) = t$. On the other hand, suppose that $w(r_1, r_2) \geq t$. Suppose also that $w(r_1, r_2) = u(y)$, $y \in X$. Hence, $f_y \gtrsim f_x$ and $(r_1, r_2) \in U_y$. Thus, $(r_1, r_2) \in U_x$.

**F.16.** If $w(r_1, r_2) > -\infty$, $r_1' > r_1$ and $r_2' > r_2$, then $w(r_1', r_2) > w(r_1, r_2)$ and

$w(r_1, r_2') \geq w(r_1, r_2)$. If $w(r_1, r_2) = -\infty$, then $w(r_1', r_2) = -\infty$ for all $r_1' \in \mathbb{R}$.

The case when $\mathcal{F}_a = \emptyset$ is easy. Suppose that $\mathcal{F}_a \neq \emptyset$. Let $r_1' > r_1$ and $r_2' > r_2$ be given. Since $w(r_1, r_2) > -\infty$, then $w(r_1, r_2) = u(x)$ for some $x \in X$. Then $(r_1, r_2) \in U_x$ and $(r_1, r_2) \notin U_y$ for any $y \in X$ such that $f_y \succ f_x$. By $F.11$, $(r_1', r_2) \in U_x$, and there exists $y \in X$ such that $f_y \succ f_x$ and $(r_1', r_2) \in U_y$. Thus, $w(r_1', r_2) > w(r_1, r_2)$. The fact that $w(r_1, r_2') \geq w(r_1, r_2)$ is obvious. If $w(r_1, r_2) = -\infty$, then $(r_1, r_2) \notin U_x$ for all $x \in X$. By $F.12$, $w(r_1', r_2) = -\infty$ for all $r_1 \in \mathbb{R}$.

**F.17.** For any $x \in X$ and $f \in F_x$, there exists $t \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$ such that $(t, v_f(t)) \in U_x$.

Fix $x \in X$ and $f \in F_x$. For all $g \in \mathcal{F}_a$ Denote by $U_g$ the set $\{(r_1, r_2) \in \mathbb{R} \times [0, 1] \mid r_1 \geq E_{\underline{l}_g} u, r_2 \geq v_f(r_1)\}$. Suppose the contrary that for all $t \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$, $(t, v_f(t)) \notin U_x$, i.e., $(t, v_f(t)) \in \mathbb{R} \times [0, 1] \setminus U_g$ for some $g \in F_x$. Note that $G_f := \{(t, v_f(t) \mid t \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]\}$ is compact and $\mathbb{R} \times [0, 1] \setminus U_g$ is open for all $g \in F_x$. Then there exist $g_1, ..., g_N \in F_x$ such that $G_f \subseteq \cup_{n=1}^{N} (\mathbb{R} \times [0, 1] \setminus U_{g_n})$. Since $v_f(E_{\underline{l}_f} u) = 1$, then $\underline{l}_{g_n} \succ \underline{l}_f$ for some $n \in \{1, ..., N\}$. For any $l \in \mathcal{L}_1$ such that $\bar{l}_f \succsim l \succ \underline{l}_f$, $v_f(E_l u) < 1$, and either $E_l u < E_{\underline{l}_{g_n}} u$ or $v_f(E_l u) < v_{g_n}(E_l u)$ for some $n \in \{1, ..., N\}$. In the former case, we also get that $v_f(E_l u) < 1 = v_{f_n}(E_l u)$. Hence, $(f_n, l) \succ' (f, l)$. By A.6, $\max\{g_n \mid n = 1, ..., N\} \succ f$, which is a contradiction.

**F.18.** The function $W$ defined in Theorem 1 is well-defined and represents $\succsim$. Moreover, $W$ is bounded in translation.

Suppose that $\mathcal{F}_a = \emptyset$. Fix $f \in \mathcal{F}_0$. Then $\underline{l}_f \sim \bar{l}_f$ and $W(f) = w(E_{\underline{l}_f} u, 1) = E_{\underline{l}_f} u$. Moreover, by $F.3$, $f \sim \underline{l}_f$. Clearly, $W$ is well-defined and represents $\succsim$. Suppose that

$\mathcal{F}_a \neq \emptyset$. If $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, then $f \sim \bar{l}_f$ and $W(f) = E_{\underline{l}_f} u$ by F.8 and F.9. If $f \in \mathcal{F}_a$, then by F.3 and F.10, there exists $x \in X$ such that $f \in F_x$. By $F.17$, there exists $t \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$ such that $(t, v_f(t)) \in U_x$, and $(t, v_f(t)) \notin U_y$ for all $y \in X$ such that $f_y \succ f_x$. Hence, $w(t, v_f(t)) = u(x)$. For any $t' \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$ such that $(t', v_f(t')) \notin U_x$, F.7 implies that $(t', v_f(t')) \notin U_y$. By F.14, $w(t', v_f(t')) < u(x)$. Hence, $W(f) = w(t, v_f(t)) = u(x)$, which implies that $W$ is well-defined and represents $\succsim$. Lastly, since $W(f) = I(u(f))$, $W$ is bounded in translation by F.4.

**F.19.** If $w' : \mathbb{R} \times [0,1] \to [-\infty, \infty)$ also satisfies the desired properties, then $w' \leq w$.

Suppose that contrary that $w(r_1, r_2) < w'(r_1, r_2)$ for some $(r_1, r_2) \in \mathbb{R} \times [0, 1]$. Let $x \in X$ be such that $u(x) = w'(r_1, r_2)$. If $\mathcal{F}_a = \emptyset$, then $u(x) > w(r_1, r_2) = r_1$. Since $w'$ is normalized and strictly increasing, then $r_2 < 1$ and $w'(u(x), 1) = u(x) > w'(r_1, r_2)$, which is a contradiction. If $\mathcal{F}_a \neq \emptyset$, then $(r_1, r_2) \notin U_x$, and thus there exists $f \in \mathcal{F}_x$ such that either $r_1 < E_{\underline{l}_f} u$ or $r_2 < v_f(r_1)$. In either case, there exists $t < 0$ such that $v_{f_t}(r_1) = r_2$ and $r_1 \in [E_{\underline{l}_{f_1}} u, E_{\bar{l}_{f_1}} u]$. Let $W' : \mathcal{L}_1 \cup \mathcal{F}_0 \to \mathbb{R}$ be the corresponding function defined as in Theorem 1 by $w'$. Thus, $W'(f_t) \geq w'(r_1, r_2) = u(x)$. Hence, $f_t \succsim f_x$, which contradicts that $f_x \sim f \succ f_t$.

Conversely, suppose (2) holds. We only check A.4.2 and A.6.

First, we check A.4.2. Since $W$ is bounded in translation, then there exists $k, k > 0$ such that for any $f \in \mathcal{F}_a$ and $t > 0$, $W(f_t) - W(f) \in [k't, kt]$. Pick $\alpha, \beta, \gamma \in (0, 1)$ such that $k' = \frac{1-\gamma}{1-\alpha}$ and $k = \frac{1-\beta}{1-\alpha}$. Fix $f \in \mathcal{F}_a$, $g \in \mathcal{F}_0 \setminus \mathcal{F}_a$ and $x, y \in X$ such that $f_y \succ f_x$.

Then $\beta g + (1-\beta) f_x \succsim \alpha f + (1-\alpha) f_x \to I(u(\beta g + (1-\beta) f_x)) \geq I(u(\alpha f + (1-\alpha) f_x)) \to \beta E_{\underline{l}_g} u + (1-\beta) u(x) \geq I(\alpha u(f) + (1-\alpha) u(x) e) \to I(u(\beta g + (1-\beta) f_y)) = \beta E_{\underline{l}_g} u + (1-\beta) u(y) = \beta E_{\underline{l}_g} u + (1-\beta) u(x) + (1-\beta)[u(y) - u(x)] \geq I(\alpha u(f) + (1-\alpha) u(x) e) + (1-\beta)[u(y) - u(x)] =$

$I(\alpha u(f)+(1-\alpha)u(x)e)+k(1-\alpha)[u(y)-u(x)] \geq I(\alpha u(f)+(1-\alpha)u(y)e) = I(u(\alpha f+(1-\alpha)f_y))$. Hence, $\beta g + (1-\beta)f_y \gtrsim \alpha f + (1-\alpha)f_y$. Similarly, one can check the other part of A.4.2.

To see A.6, first fix $f_1, f_2 \in \mathcal{F}_0$ such that $(f_1, l) \gtrsim '(f_2, l)$ for all $l \in \mathcal{L}_1$. For all $t \in \mathbb{R}$, $v_{f_1}(t) \geq v_{f_2}(t)$, and thus $w(t, v_{f_1}(t)) \geq w(t, v_{f_2}(t))$. Hence, $E_{\bar{l}_{f_1}} u \geq E_{\bar{l}_{f_2}} u$. If $t \in [E_{\underline{l}_{f_2}} u, E_{\bar{l}_{f_2}} u] \setminus [E_{\underline{l}_{f_1}} u, E_{\bar{l}_{f_1}} u]$, then $t < E_{\underline{l}_{f_1}} u$, and thus $w(E_{\underline{l}_{f_1}} u, 1) > w(t, v_{f_1}(t)) \geq w(t, v_{f_2}(t))$. Hence, for all $t \in [E_{\underline{l}_{f_2}} u, E_{\bar{l}_{f_2}} u]$, there exists $t' \in [E_{\underline{l}_{f_1}} u, E_{\bar{l}_{f_1}} u]$ such that $w(t', v_{f_1}(t')) \geq w(t, v_{f_2}(t))$. Thus, $W(f_1) \geq W(f_2)$, i.e., $f_1 \gtrsim f_2$.

Next fix $N \in \mathbb{Z}_+$ and $f, f_1, ..., f_N \in \mathcal{F}_a$ such that $\max\{\underline{l}_{f_n} \mid n = 1, ..., N\} \succ \underline{l}_f$ and $\max\{(f_n, l) \mid i = 1, ..., N\} \succ' (f, l)$ for all $l$ satisfying $\bar{l}_f \gtrsim l \succ \underline{l}_f$. Suppose that $W(f) = w(t, v_f(t))$, $t \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]$. If $t = E_{\underline{l}_f} u$, then $t < E_{\underline{l}_{f_n}} u$ for some $n \in \{1, ..., N\}$. Thus, $W(f) = t < E_{\underline{l}_{f_n}} u \leq W(f_n)$. If $t \in (E_{\underline{l}_f} u, E_{\bar{l}_f} u]$, then $v_f(t) < v_{f_n}(t)$ for some $n \in \{1, ..., N\}$. It is easy to see that there exists $t' \in (E_{\underline{l}_{f_n}} u, E_{\bar{l}_{f_n}} u)$ such that $(t', v_{f_n}(t')) > (t, v_f(t))$. Hence, $W(f_n) > W(f)$ as desired. $\square$

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